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Formal stability of circular vortices

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The second variation of a linear combination of energy and angular momentum is used to investigate the formal stability of circular vortices. The analysis proceeds entirely in terms of Lagrangian displacements to overcome problems that otherwise arise when one attempts to use Arnol'd's Eulerian formalism. Specific attention is paid to the simplest possible model of an isolated vortex consisting of a core of constant vorticity surrounded by a ring of oppositely signed vorticity. We prove that the linear stability regime for this vortex coincides with the formal stability regime. The fact that there are formally stable isolated vortices could imply that there are provable nonlinearly stable isolated vortices. The method can be applied to more complicated vortices consisting of many nested rings of piecewise-constant vorticity. The equivalent expressions for continuous vorticity distributions are also derived.

1. Introduction

In this paper we use classical calculus of variations to establish criteria for formal and linear stability of planar circular vortices with piecewise-constant and piecewise-continuous vorticity in an ideal, unbounded and incompressible fluid. Linear stability is typically investigated by a normal-modes analysis of the linearized equations. With such an analysis Rayleigh's inflexion-point theorem can be derived. This theorem states that a necessary condition for instability is that the vorticity gradient changes sign somewhere in the vortex (Drazin & Reid 1981). Thus, a sufficient condition for linear stability is that the vorticity gradient does not change sign anywhere. However, not all vortices with a vorticity gradient that changes sign are linearly unstable. The occurrence of an inflexion point is necessary but not sufficient for instability. Observations (Kloosterziel & van Heijst 1991) and several numerical and analytical studies indicate that for a vortex to become unstable the velocity profile has to fall off to zero in the outer region sufficiently rapidly.

For instance, Flierl (1988) has solved the normal-modes equations analytically for a class of isolated model vortices. These vortices consist of a core of constant vorticity $q_1 = 1$ within the non-dimensional radius $r = 1$ plus an annulus of oppositely signed vorticity $q_2 = -q < 0$ between $r = 1$ and $r = d$ (see figure 2). These vortices all have vanishing circulation at $r = d$. For large enough d (small q) they are linearly stable to perturbations at all wavenumbers. Here the wavenumber is defined in the usual way with the angular dependence of the perturbation in polar coordinates taken to be proportional to $\exp(il\theta)$. For $d < 2\ (q > \frac{1}{3})$, $l = 2$ perturbations are unstable; for $d < (1+\sqrt{2})^{\frac{1}{2}}\ (q > 1/\sqrt{2})$, the $l = 3$ modes are unstable, and so on.

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Calculation no.	$q(r)$	Growing wavenumber	$q(r \to \infty)$
(i)	$(1-\frac{1}{2}r^2)(1+r^2)^{-\frac{5}{2}}$	stable	$q \propto r^{-3}$
(i)	$(1-r^2)(1+r^2)^{-3}$	stable	$q \propto r^{-4}$
(iii)	$(1-2r^2)(1+r^2)^{-4}$	stable	$q \propto r^{-6}$
(iv)	$(1-\frac{1}{2}r^2)(\exp(-\frac{1}{2}r^2))$	$l=2$	$q \propto r^2 \mathrm{e}^{-\frac{1}{2}r^2}$

TABLE 1. Some of the linear stability properties of families of vortices investigated by Gent & McWilliams (1986) by means of a finite-differences method

The steeper the velocity profile (solid line in figure 2a), which translates into a larger amplitude of the vorticity of the annulus relative to that of the core (solid line in figure 2b), the higher is the wavenumber of the fastest growing mode, and for a shallow enough profile ($q < \frac{1}{3}$), the vortices are linearly stable.

Carton & McWilliams (1989) have studied the stability of the one-parameter family of continuous vorticity profiles

$$q_\alpha(r) = (1-\tfrac{1}{2}\alpha r^\alpha)\exp(-r^\alpha)$$

(with $\alpha > 0$). Each of these profiles has one inflexion point. For increasing values of α these profiles become steeper, and Carton & McWilliams found through normal-modes analysis (numerically) that for α-values smaller than 1.9 the vortices are linearly stable, whereas for larger values the vortices are unstable.

Gent & McWilliams (1986) solved the normal-modes equations numerically for several different vorticity profiles with an inflexion point and some of their results are summarized in table 1. The vorticity profiles are shown in the second from the left column and the asymptotic behaviour in the rightmost column; the latter indicates the steepness of the profile. All the profiles have maximum vorticity at the centre and a single negative minimum, which corresponds to the single inflexion point of the velocity profile, but only the very steep profile with exponential decay of vorticity is linearly unstable. All the above-mentioned model vortices are isolated and serve as models for typical oceanic vortices. Such isolated vortices will be the main topic of the present paper.

These observations raise the question of how to discern between the unstable and stable cases among the vortices that satisfy Rayleigh's criterion for instability. It is interesting to note that the popular Arnol'd (1966) method fails to discern between the cases shown in table 1 (see Appendix A). We have therefore developed a purely Lagrangian method to overcome the problem. The method is, as is Arnol'd's, essentially an 'energy method' aimed at proving Liapunov stability with the aid of the conserved quantities of the equations of motion. As a first step to investigating nonlinear stability, i.e. stability to finite perturbations, we formulate in the present paper criteria for formal stability. A stationary flow is called formally stable (see Holm *et al.* 1985) if there is a conserved quantity such that the first variation of this quantity (i.e. the lowest-order change due to arbitrary infinitesimal perturbations) is zero whereas the second variation is positive or negative definite. Formal stability implies linear stability because the second variation of this conserved quantity is invariant in the linearized dynamics. In finite-dimensional systems formal stability implies nonlinear stability whereas in infinite dimensions it is a necessary prerequisite for nonlinear stability.

The object of the present paper is to derive criteria for formal stability for planar, circular vortices in an unbounded ideal fluid. For this we derive expressions for the second variations of energy and angular momentum under vorticity-preserving perturbations. We do not specify local perturbations on the field of interest, say the vorticity, as a function of the spatial coordinates but instead investigate the changes in energy and angular momentum due to prescribed displacements of individual fluid elements. This makes the mathematics more complicated than the Eulerian approach of Arnol'd but this appears to be necessary for isolated vortices. With such a Lagrangian approach, Dritschel (1988) has been able to prove the nonlinear stability of a vortex patch and of vortices with monotonically decreasing vorticity. For such vortices only angular momentum and area conservation need be considered. We have found that for isolated vortices and other vortices with non-monotonically decreasing vorticity (formal and linear), stability can only be proven if kinetic energy is added to the analysis. The calculation of the variations of energy due to Lagrangian displacements of vorticity contours is much harder than for angular momentum, but this is an important novel ingredient of the present paper.

It can be shown (see §2.1), that the first variations of energy and angular momentum vanish for circular vortices. If the second-order variations of some linear combination of the energy and angular momentum are sign definite, then we have formal stability. The second variations are the lowest-order terms of a Taylor series expansion around a stationary state for arbitrary perturbations; therefore, if formal stability can be proven, there is reason to hope for stability to finite-amplitude perturbations.

An overview of the contents of this paper is the following. Arnol'd (1965) showed that any stationary flow (with arbitrary vorticity) provides an extremum in energy with respect to isocirculational variations (i.e. divergenceless variations that leave the circulation along all material curves unchanged). For a vortex with constant vorticity in a single bounded domain the isocirculational variations are simply all area-preserving variations. This is also true for vortices with multiple regions of constant vorticity; for such vortices the isocirculational variations are those perturbations on the closed curves bounding the regions that leave the area of each region unchanged. As a special cae of Arnol'd's result we verify in §2.1 that any stationary flow with constant vorticity in a single closed domain on $\boldsymbol{R}^2$ has extremal energy with respect to area-preserving variations (i.e. the first variation in energy is zero for such variations).

We proceed in §2.1 with establishing the fact that only the *circular* vortex patch provides an extremum in angular momentum. Definiteness of the second variation of the angular momentum therefore implies the formal stability of the circular patch. We show in §2.2 that the circular vortex patch locally minimizes angular momentum and in §2.3 that it locally maximizes energy. Both results prove the formal stability of the circular patch.

In §3 we derive the expressions for the second variations of energy and angular momentum under area-preserving variations for circular vortices consisting of n nested concentric rings of piecewise-constant vorticity. These results form the basis for further research into formal stability properties of such vortices. To illustrate their use we focus attention in §4 on the simplest possible model of an isolated vortex which consists of $n = 2$ rings. In §4.1 we show that the linear stability regimes for this model can be uncovered by considering the sign of the second variation of energy on a manifold on which the second variation of angular momentum vanishes. In §4.2 we show how to prove formal stability with the aid of the second variations; in particular we show that the linear stability regime of the above-mentioned model vortex coincides with the formal stability regime.

Finally, in §5 we point out further possible generalizations of the formalism to the case of vortices with continuous vorticity and discuss the possibility of extending the analysis to higher order.

2. Single vortex patches

In this section we consider a flow that has constant vorticity q in a simply connected domain with boundary Γ, outside of which the flow is irrotational (a vortex 'patch'). Let the boundary of the vortex patch on $\boldsymbol{R}^2$ be given as the closed curve Γ. The area $A(\Gamma)$ of a domain $\mathscr{D}$ enclosed by Γ is determined as follows (see figure 1). Consider a position vector $\boldsymbol{r}$ on $\boldsymbol{R}^2$ (i.e. $\boldsymbol{r} = (x, y)$ with x and y Cartesian coordinates). The (oriented) area swept out by a small increment $\mathrm{d}\boldsymbol{r}$ is $\frac{1}{2}\boldsymbol{r} \wedge \mathrm{d}\boldsymbol{r}$. Now let Γ be parametrized on $\boldsymbol{R}^2$ according to

$$\Gamma\colon \quad \{x, y\} = \{s_x(\theta), s_y(\theta)\}, \quad \theta \in [\theta_1, \theta_2], \tag{1}$$

with $s_x(\theta_1) = s_x(\theta_2)$ and $s_y(\theta_1) = s_y(\theta_2)$. The area A of $\mathscr{D}$ is then

$$A(\Gamma) = \tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \boldsymbol{s} \wedge \dot{\boldsymbol{s}}\, \mathrm{d}\theta, \tag{2}$$

where $\boldsymbol{s} = \boldsymbol{i}s_x + \boldsymbol{j}s_y$. A dot over a dependent variable denotes differentiation with respect to θ here and $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are three unit vectors spanning a positively oriented Cartesian coordinate system in $\boldsymbol{R}^3$.

A well-known result from fluid mechanics is that for an ideal incompressible fluid, the area enclosed by Γ is a constant of motion. From the equations of motion it can further be deduced that kinetic energy E, angular momentum L and linear impulse $\boldsymbol{P}$ are also invariants. The relevant invariant parts of the kinetic energy, angular momentum and linear impulse that depend on the vorticity distribution $q(x, y)$ are given by the following integrals (Batchelor 1967):

$$E = \frac{1}{2}\iint_{\mathscr{D}} q\psi\, \mathrm{d}x\, \mathrm{d}y, \tag{3}$$

$$L = \iint_{\mathscr{D}} (x^2 + y^2)\, q\, \mathrm{d}x\, \mathrm{d}y, \tag{4}$$

$$\boldsymbol{P} = \{P_x, P_y\} = \left\{\iint_{\mathscr{D}} yq\, \mathrm{d}x\, \mathrm{d}y,\ -\iint_{\mathscr{D}} xq\, \mathrm{d}x\, \mathrm{d}y\right\}, \tag{5}$$

where the stream function ψ is related to the vorticity distribution according to

$$\psi(\boldsymbol{r}) = -\frac{1}{2\pi}\iint_{\mathscr{D}} q(\boldsymbol{r}')\log|\boldsymbol{r} - \boldsymbol{r}'|\, \mathrm{d}x'\, \mathrm{d}y'. \tag{6}$$

With this definition q and ψ are related according to $q = -\nabla^2\psi$. Since the domain is only determined by Γ, we also write $E = E(\Gamma)$, $L = L(\Gamma)$ and $\boldsymbol{P} = \boldsymbol{P}(\Gamma)$.

We wish to determine the changes in the conserved quantities if the boundary of the vortex patch is slightly perturbed. The magnitude of the perturbation is measured by a non-dimensional small parameter ϵ and the variations are ordered in different powers of ϵ. The $O(\epsilon)$ change is called the first variation, the $O(\epsilon^2)$ change the second variation, etc. It is of interest sometimes to determine the changes in one of the invariants while keeping another one fixed. In particular we are interested in the first and second variations under area-preserving perturbations.

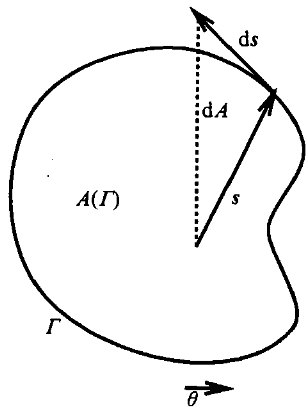

FIGURE 1. Diagram showing a closed curve Γ parametrized with a vector $\boldsymbol{s}(\theta)$, where θ is a coordinate running along the curve Γ. The oriented area dA swept out by a small increment $\mathrm{d}\boldsymbol{s} = \dot{\boldsymbol{s}}\,\mathrm{d}\theta$ is equal to $\frac{1}{2}\boldsymbol{s} \wedge \mathrm{d}\boldsymbol{s}$.

It is tempting to describe the perturbation as follows. Let Γ be changed to $\Gamma + \epsilon\delta\Gamma$ according to the parametrization

$$\Gamma + \epsilon\delta\Gamma\colon \quad \{x, y\} = \{s_x(\theta) + \epsilon u_x(\theta), s_y(\theta) + \epsilon u_y(\theta)\}, \quad \theta \in [\theta_1, \theta_2],$$

where ϵ is an arbitrary constant and $u_x(\theta)$ and $u_y(\theta)$ are arbitrary functions of θ which are independent of ϵ ('weak variations'). Periodicity in θ and certain continuity requirements are assumed to be fulfilled here. Special care needs to be taken if one wants area to be conserved. For instance, consider a circle of radius 1. The boundary is perturbed to $r(\theta) = 1 + \epsilon\delta r(\theta)$ (we have polar coordinates (r, θ) in mind here). The change in area is

$$\epsilon \int_0^{2\pi} \delta r(\theta)\,\mathrm{d}\theta + \tfrac{1}{2}\epsilon^2 \int_0^{2\pi} \delta r(\theta)^2\,\mathrm{d}\theta.$$

So although clearly at $O(\epsilon)$ area conservation can be satisfied, it cannot at $O(\epsilon^2)$. To overcome this problem we need to introduce a more general form of perturbation. In order to satisfy area conservation at all orders $\boldsymbol{s}$ is changed to $\boldsymbol{s} + \epsilon\boldsymbol{u}_1 + \frac{1}{2}\epsilon^2\boldsymbol{u}_2 + O(\epsilon^3)$, with $|\boldsymbol{u}_i|$ an $O(1)$ function, independent of ϵ. Substitution shows that the area change is

$$\begin{aligned}\Delta A &= A(\Gamma + \delta\Gamma) - A(\Gamma)\\ &= \epsilon\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}}\,\mathrm{d}\theta + \tfrac{1}{2}\epsilon^2\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \{\boldsymbol{u}_1 \wedge \dot{\boldsymbol{u}}_1 + \boldsymbol{u}_2 \wedge \dot{\boldsymbol{s}}\}\,\mathrm{d}\theta + O(\epsilon^3). \qquad (7)\end{aligned}$$

For given $\boldsymbol{u}_1$ we can always find a $\boldsymbol{u}_2$ such that at $O(\epsilon^2)$ area is conserved (and by an appropriate choice of the $\boldsymbol{u}_i$ at any order area can be conserved).

An example may clarify this. Take for Γ a circle of radius 1. Perturb Γ to (in polar coordinates):

$$\Gamma + \delta\Gamma\colon \quad r(\theta) = 1 + \epsilon\delta r_1(\theta) + \tfrac{1}{2}\epsilon^2\delta r_2(\theta) + \dots. \qquad (8)$$

In this case the parameter θ is just the polar angle, running from 0 to 2π. The change in area is then

$$\Delta A = \epsilon \int_0^{2\pi} \delta r_1(\theta)\,\mathrm{d}\theta + \tfrac{1}{2}\epsilon^2 \int_0^{2\pi} (\delta r_1^2(\theta) + \delta r_2(\theta))\,\mathrm{d}\theta + \dots,$$

so area is conserved at all orders by taking

$$\int_0^{2\pi} \delta r_1(\theta)\,\mathrm{d}\theta = 0, \quad \int_0^{2\pi} \delta r_2(\theta)\,\mathrm{d}\theta = -\int_0^{2\pi} \delta r_1^2(\theta)\,\mathrm{d}\theta, \text{ etc.}$$

We introduce the following notation:

$$A(\Gamma+\delta\Gamma) = A_0+\epsilon A_1+\tfrac{1}{2}\epsilon^2 A_2+\dots,$$

where $$A_1 = \frac{\partial A(\Gamma+\delta\Gamma)}{\partial\epsilon}\bigg|_{\epsilon=0}, \quad A_2 = \frac{\partial^2 A(\Gamma+\delta\Gamma)}{\partial\epsilon^2}\bigg|_{\epsilon=0}, \text{ etc.}$$

and $A_0 = A(\Gamma)$. With (7) we get

$$A_1 = \int_{\theta_1}^{\theta_2} \mathrm{d}A_1(\theta), \quad \mathrm{d}A_1(\theta) = \boldsymbol{k}\cdot\boldsymbol{u}_1\wedge\dot{\boldsymbol{s}}\,\mathrm{d}\theta, \tag{9}$$

$$A_2 = \int_{\theta_1}^{\theta_2} \mathrm{d}A_2(\theta), \quad \mathrm{d}A_2(\theta) = \{\boldsymbol{k}\cdot\boldsymbol{u}_1\wedge\dot{\boldsymbol{u}}_1+\boldsymbol{k}\cdot\boldsymbol{u}_2\wedge\dot{\boldsymbol{s}}\}\,\mathrm{d}\theta. \tag{10}$$

Below, the case where $\boldsymbol{s}$ describes a circle will be of particular interest. In that case one has $\boldsymbol{u}_1\cdot\boldsymbol{s} = \boldsymbol{k}\cdot\{\boldsymbol{u}_1\wedge\dot{\boldsymbol{s}}\}$. Furthermore, it is customary to represent a perturbation to the circle as the angle-dependent departure from circular symmetry, i.e. by giving the perturbed boundary as (see (8))

$$r(\theta) = r_0+\epsilon\delta r(\theta)+O(\epsilon^2). \tag{11}$$

One way of relating δr to a $\boldsymbol{u}_1$ field is by simply taking

$$u_{1,x}(\theta) = \delta r(\theta)\cos\theta, \quad u_{1,y}(\theta) = \delta r(\theta)\sin\theta, \tag{12}$$

which amounts to choosing $\boldsymbol{u}_1$ parallel to $\boldsymbol{s}$. With this representation we find

$$\boldsymbol{u}_1\cdot\boldsymbol{s}(\theta) = r_0\,\delta r(\theta)\cos^2\theta+r_0\,\delta r(\theta)\sin^2\theta = r_0\,\delta r(\theta). \tag{13}$$

It will be convenient to simplify the notation further, and we define

$$\phi(\theta) = \boldsymbol{u}_1\cdot\boldsymbol{s}(\theta) = r_0\,\delta r(\theta). \tag{14}$$

In the case that Γ is a circle we thus have for $\mathrm{d}A_1$ (as given by (9)):

$$\mathrm{d}A_1(\theta) = \phi(\theta)\,\mathrm{d}\theta. \tag{15}$$

Referring to (3)–(6) we see that in order to determine the different variations, we need to know how integrals of type

$$I(\Gamma) = \iint_{\mathscr{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

change when the boundary Γ of the domain $\mathscr{D}$ is perturbed. With Green's theorem we find (see Appendix B):

$$I_1 = \frac{\mathrm{d}I}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} = \int_{\theta_1}^{\theta_2} f(\boldsymbol{s})\,\mathrm{d}A_1(\theta), \tag{16}$$

$$I_2 = \frac{\mathrm{d}^2 I}{\mathrm{d}\epsilon^2}\bigg|_{\epsilon=0} = \int_{\theta_1}^{\theta_2} \{\boldsymbol{u}_1\cdot\nabla f(\boldsymbol{r}=\boldsymbol{s})\}\,\mathrm{d}A_1(\theta)+\int_{\theta_1}^{\theta_2} f(\boldsymbol{s})\,\mathrm{d}A_2(\theta), \tag{17}$$

where the $\mathrm{d}A_i$ are given by (9) and (10). The higher-order variations will not be needed in this paper. Again, if s is a circle, (14) and (15) apply, and with (12) we have in that case

$$\boldsymbol{u}_1 \cdot \nabla f = \delta r \frac{\partial f}{\partial r}.$$

We write

$$\left.\begin{aligned} \psi &= \psi_0 + \epsilon\psi_1 + \tfrac{1}{2}\epsilon^2\psi_2 + \dots, & E &= E_0 + \epsilon E_1 + \tfrac{1}{2}\epsilon^2 E_2 + \dots, \\ L &= L_0 + \epsilon L_1 + \tfrac{1}{2}\epsilon^2 L_2 + \dots, & P &= P_0 + \epsilon P_1 + \tfrac{1}{2}\epsilon^2 P_2 + \dots, \end{aligned}\right\} \tag{18}$$

where the subscript 0 denotes the unperturbed basis state, the subscript 1 the $O(\epsilon)$ change due to a slight perturbation of the boundary, defined by (16), and so on.

Thus the first and second variations of the stream function are

$$\psi_1(\boldsymbol{r}) = -\frac{q}{2\pi}\int_{\theta_1}^{\theta_2} \log|\boldsymbol{r}-\boldsymbol{s}(\theta)|\,\mathrm{d}A_1(\theta), \tag{19}$$

$$\psi_2(\boldsymbol{r}) = -\frac{q}{2\pi}\int_{\theta_1}^{\theta_2} \boldsymbol{u}_1 \cdot \nabla' \log|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}A_1(\theta) - \frac{q}{2\pi}\int_{\theta_1}^{\theta_2} \log|\boldsymbol{r}-\boldsymbol{s}(\theta)|\,\mathrm{d}A_2(\theta), \tag{20}$$

where
$$\nabla' = \boldsymbol{i}\frac{\partial}{\partial x'} + \boldsymbol{j}\frac{\partial}{\partial y'}.$$

The expression in (20) is understood to be evaluated at $\boldsymbol{r}' = \boldsymbol{s}(\theta)$. The first two angular momentum variations are

$$L_1 = q\int_{\theta_1}^{\theta_2} |\boldsymbol{s}(\theta)|^2\,\mathrm{d}A_1(\theta), \tag{21}$$

$$L_2 = q\int_{\theta_1}^{\theta_2} 2\boldsymbol{u}_1 \cdot \boldsymbol{s}(\theta)\,\mathrm{d}A_1(\theta) + q\int_{\theta_1}^{\theta_2} |\boldsymbol{s}(\theta)|^2\,\mathrm{d}A_2(\theta). \tag{22}$$

The energy functional is slightly more complicated. When the boundary is changed, the change in energy is

$$\begin{aligned} \Delta E &= q\iint_{\mathscr{D}+\delta\mathscr{D}} (\psi_0 + \epsilon\psi_1 + \tfrac{1}{2}\epsilon^2\psi_2 + \dots)\,\mathrm{d}x\,\mathrm{d}y - q\iint_{\mathscr{D}} \psi_0\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathscr{D}+\delta\mathscr{D}} \psi_0\,\mathrm{d}x\,\mathrm{d}y - q\iint_{\mathscr{D}} \psi_0\,\mathrm{d}x\,\mathrm{d}y + q\iint_{\mathscr{D}+\delta\mathscr{D}} (\epsilon\psi_1 + \tfrac{1}{2}\epsilon^2\psi_2 + \dots)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{23}$$

By expanding the integrals in (23) the following expressions are derived for the first two variations of the energy:

$$E_1 = q\int_{\theta_1}^{\theta_2} \psi_0(\boldsymbol{s})\,\mathrm{d}A_1(\theta) + q\iint_{\mathscr{D}} \psi_1\,\mathrm{d}x\,\mathrm{d}y, \tag{24}$$

$$E_2 = q\int_{\theta_1}^{\theta_2} \boldsymbol{u}_1 \cdot \nabla\psi_0(\theta)\,\mathrm{d}A_1(\theta) + q\int_{\theta_1}^{\theta_2} \psi_0(\boldsymbol{s})\,\mathrm{d}A_2(\theta) + q\int_{\theta_1}^{\theta_2} 2\psi_1(\boldsymbol{s})\,\mathrm{d}A_1(\theta) + q\iint_{\mathscr{D}} \psi_2\,\mathrm{d}x\,\mathrm{d}y. \tag{25}$$

The variations of the x-component of the linear impulse are

$$P_{x,1} = q\int_{\theta_1}^{\theta_2} s_y(\theta)\,\mathrm{d}A_1(\theta), \tag{26}$$

$$P_{x,2} = q\int_{\theta_1}^{\theta_2} u_{1,y}(\theta)\,\mathrm{d}A_1(\theta) + q\int_{\theta_1}^{\theta_2} s_y(\theta)\,\mathrm{d}A_2(\theta), \tag{27}$$

with similar expressions for the variations of P_y. Here $u_{1,y}$ denotes the y-component of the vector $\boldsymbol{u}_1$.

2.1. *Conditional extrema for single vortex patches*

Let us assume that the flow is stationary in some (possibly translating) reference frame. Then according to Arnol'd (1965) under area-preserving variations (isocirculational variations) the first variation of the energy vanishes. This is easily verified as follows. Necessarily we have that the boundary Γ is a streamline, so the stream function is constant on Γ, i.e.

$$\psi_0(\Gamma) = \text{constant}. \tag{28}$$

With this assumption the first part of the first variation of the energy, as given by (24), is seen to be equal to $q\psi_0(\Gamma)A_1$, where A_1 is the $O(\epsilon)$ area change given by (9). But, by substitution of (19) in (24) and interchanging the order of integration, it is found that the second part of (24) is also equal to $q\psi_0(\Gamma)A_1$ and we therefore have the following result:

$$E_1 = 2q\psi_0(\Gamma)A_1. \tag{29}$$

This implies that when the variations are area-preserving (i.e. $A_1 = A_2 = 0$), the energy does not change at lowest order. So, any stationary flow consisting of a single constant-vorticity patch on $\boldsymbol{R}^2$ has an extremum in energy with respect to area-preserving variations.

A particular flow is singled out, however, if angular momentum is added as a constraint. To show this we construct a functional F

$$F(\Gamma) = L(\Gamma) + \lambda q A(\Gamma) + \tfrac{1}{2}\mu E(\Gamma), \tag{30}$$

where λ and μ are Lagrange multipliers (constants). Under which circumstances does the first variation vanish, i.e.

$$\left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = F_1 = 0\,?$$

Clearly we have $F_1 = L_1 + \lambda q A_1 + \frac{1}{2}\mu E_1$, which by substitution of (21) and (29) is equal to

$$F_1 = q\int_{\theta_1}^{\theta_2} \{|\boldsymbol{s}|^2 + \lambda + \mu\psi_0(\boldsymbol{s})\}\,\mathrm{d}A_1(\theta). \tag{31}$$

Since $\mathrm{d}A_1(\theta)$ is arbitrary (through the perturbative vector field $\boldsymbol{u}_1(\theta)$), we find the following Euler equation (du Bois Reymond's lemma; see Ewing 1985):

$$|\boldsymbol{s}|^2 = -\lambda - \mu\psi(\Gamma). \tag{32}$$

This solution tells us that Γ is a circle, which corresponds to a valid stationary solution. Without the energy constraint (put $\mu = 0$) we also get this result. The circular vortex patch yields an extremum in angular momentum for fixed vortex area. For fixed area the energy is invariant at lowest-order and it does not therefore add any true constraints (only if the basic state is stationary and only in the first variation).

Consider now the additional constraint of linear impulse. We need not use both P_x and P_y because by a simple coordinate change (a rotation) one of the two can be made zero. We take P_x and construct the functional

$$F(\Gamma) = L(\Gamma) + \lambda q A(\Gamma) + \tfrac{1}{2}\mu E(\Gamma) + \alpha P_x(\Gamma). \tag{33}$$

The Euler equation is (use (26))

$$(s_x^2 + s_y^2) + \lambda + \mu\psi(\Gamma) + \alpha s_y = 0,$$

or

$$s_x^2 + (s_y + \tfrac{1}{2}\alpha)^2 = -\lambda - \mu\psi(\Gamma) + \tfrac{1}{4}\alpha^2. \tag{34}$$

This corresponds to a circle with its centre at $\{x, y\} = \{0, -\tfrac{1}{2}\alpha\}$. Thus the additional constraint of linear impulse invariance only appears as a shift of origin.

2.2. *Angular momentum*

The circular vortex patch will be the object of further study here. We begin with the second variation of the angular momentum, i.e. L_2 as given by (22). Area conservation at $O(\epsilon^2)$ implies

$$\int_{\theta_1}^{\theta_2} \mathrm{d}A_2(\theta) = 0,$$

and, since $|s|^2 =$ constant for the circular vortex patch, the second part of the integral in (22) is zero.

By substitution of (9) and noting that if $s(\theta)$ describes a circle (15) applies, with ϕ defined by (14), the second variation is

$$L_2 = 2q \int_0^{2\pi} |\boldsymbol{u}_1 \cdot \boldsymbol{s}|^2 \, \mathrm{d}\theta = 2q \int_0^{2\pi} \phi^2 \, \mathrm{d}\theta. \tag{35}$$

Because this expression is positive definite for any perturbative vector field $\boldsymbol{u}_1$, we conclude that the circular vortex locally minimizes angular momentum (for positive q) and is therefore formally stable. The stronger nonlinear stability of the circular vortex patch has been proved by Dritschel (1988) also by essentially using the angular momentum invariant and the area constraint.

2.3. *Energy*

We can also show that the circular vortex patch locally maximizes energy. Since we have already shown that the patch is formally stable, this additional information is not needed. However, for the more complicated cases of multiple vorticities, to be discussed below, the second variation of energy needs to be determined. This involves a lengthy calculation along similar lines as for a single patch and to show how it is done we discuss in detail the simpler case of a single patch here and do not go into details when discussing the more complicated case of several patches.

To start we note that in (25) the integral $\iint_{\mathscr{D}} \psi_2 \, \mathrm{d}x \, \mathrm{d}y$ is equal to the sum of the first two integrals appearing in (25). This follows by substitution of (20) and taking the $\int \mathrm{d}\theta$ integral outside the area integral. Moreover, if the circular boundary is denoted in polar coordinates (r, θ) by $r = r_0 =$ constant, we have

$$\boldsymbol{u}_1 \cdot \nabla\psi_0(r = r_0) = -\frac{v(r = r_0)}{r_0} \boldsymbol{u}_1 \cdot \boldsymbol{s}, \tag{36}$$

where $v(r=r_0)$ is the tangential velocity at the boundary of the vortex patch. By substitution of (36) and (19) in (25), and using (15) and (14), we obtain the following expression for the second variation of the energy:

$$E_2 = -2q\frac{v(r=r_0)}{r_0}\int_0^{2\pi}\phi^2\,\mathrm{d}\theta + 2q\psi_0(\Gamma)\int_0^{2\pi}\mathrm{d}A_2(\theta) - q^2\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\log|s(\theta) - s(\theta')|\,\phi(\theta')\,\phi(\theta)\,\mathrm{d}\theta'\,\mathrm{d}\theta. \quad (37)$$

Because of the imposed second-order area conservation ($A_2 = \int \mathrm{d}A_2 = 0$), the second contribution to this expression is zero.

The third term in (37) contains a linear integral operator $\mathscr{L}$ defined by

$$\mathscr{L}\phi(\theta) = \frac{1}{\pi}\int_0^{2\pi}\log|s(\theta)-s(\theta')|\,\phi(\theta')\,\mathrm{d}\theta', \quad (38)$$

where ϕ is defined by (14). This transform contains a symmetric Fredholm kernel of convolution type for which the eigenvalues are real and negative, and with eigenfunctions $\cos n\theta$ and $\sin n\theta$ (with $n \in N^+$). To see this we note that in complex notation the circle is represented by $s(\theta) = r_0\,\mathrm{e}^{\mathrm{i}\theta}$. Further we note that

$$\log|r_0\,\mathrm{e}^{\mathrm{i}\theta} - r_0\,\mathrm{e}^{\mathrm{i}\theta'}| = \log|r_0| + \log|1-\mathrm{e}^{\mathrm{i}(\theta'-\theta)}|.$$

By writing $\tilde{\theta} = \theta' - \theta$ and introducing the notation $\phi_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}$, we get

$$\mathscr{L}\phi_n(\theta) = \frac{1}{\pi}\int_{-\theta}^{2\pi-\theta}\log|1-\mathrm{e}^{\mathrm{i}\tilde{\theta}}|\,\mathrm{e}^{\mathrm{i}n\tilde{\theta}}\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\tilde{\theta} = \lambda_n\,\phi_n(\theta), \quad (39)$$

where the eigenvalues are

$$\lambda_n = \frac{1}{\pi}\int_{-\theta}^{2\pi-\theta}\log|1-\mathrm{e}^{\mathrm{i}\tilde{\theta}}|\,\mathrm{e}^{\mathrm{i}n\tilde{\theta}}\,\mathrm{d}\tilde{\theta}. \quad (40)$$

We have used here the imposed $O(\epsilon)$ area conservation

$$\log|r_0|\int_{\theta_1}^{\theta_2}\phi(\theta')\,\mathrm{d}\theta' = \log|r_0|\int_{\theta_1}^{\theta_2}\mathrm{d}A_1(\theta') = 0.$$

By expanding the logarithmic kernel

$$\log|1-z| = \mathrm{Re}\left\{-\sum_{m=1}^{\infty}\frac{z^m}{m}\right\}$$

we get

$$\lambda_n = \frac{-1}{\pi}\int\left\{\sum_{m=1}^{\infty}\frac{\cos m\tilde{\theta}}{m}\right\}\mathrm{e}^{\mathrm{i}n\tilde{\theta}}\,\mathrm{d}\tilde{\theta} = \frac{-1}{n}. \quad (41)$$

Since the trigonometric functions are complete in $L^2[0,2\pi]$ it is necessary to restrict the class of perturbations to this real Hilbert space, i.e. the perturbation δr has to be such that

$$\int_0^{2\pi}|\delta r(\theta)|^2\,\mathrm{d}\theta < \infty.$$

We then can develop δr in a Fourier series and calculate the inner products appearing in (37). This does not restrict the class of allowed perturbations we were considering so far. First of all it must be remembered that a perturbation should not break the circle, which implies that the displacement δr has to be continuous in θ. Moreover, the weak variations have at most an $O(1)$ amplitude so they are certainly square-integrable. The introduction of the L^2-space does therefore not restrict the original class of weak variations to a smaller set.

At this point we need the Fourier series decomposition for δr, i.e.

$$\delta r = \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta, \tag{42}$$

where in view of the constraint of area conservation no $k = 0$ component has been allowed (an $a_0, b_0 \neq 0$ corresponds to an expansion or contraction of the circle which violates area conservation at $O(\epsilon)$). By substitution in (37) we obtain

$$E_2 = -2q\frac{v(r_0)}{r_0}\sum_{k=1}^{\infty} \pi\{a_k^2+b_k^2\}\, r_0^2 + q^2 r_0^2 \sum_{k=1}^{\infty}\frac{\pi}{k}\{a_k^2+b_k^2\} = -\pi q^2 r_0^2 \sum_{k=1}^{\infty}\left(\frac{k-1}{k}\right)\{a_k^2+b_k^2\}, \tag{43}$$

where we have used $v(r_0) = \frac{1}{2}qr_0$.

It is clear that E_2 is negative definite if we exclude wavenumber-1 perturbations. Such a perturbation corresponds to an overall displacement of the vortex, and as expected this does not change the energy. This proves that the circular vortex provides a local maximum in energy for all (infinitesimal) square-integrable perturbations modulo translations.

3. Multiple vortex patches

We consider two closed curves Γ_1 and Γ_2, with Γ_2 enclosing Γ_1. The area enclosed by the inner curve Γ_1 is called $\mathscr{D}_1$ and has constant vorticity q_1. The area between Γ_1 and Γ_2 is denoted by $\mathscr{D}_2$ and has constant vorticity q_2. Furthermore, the flow exterior to Γ_2 is irrotational. In vector notation the curve Γ_1 is represented by the parametrized vector $\boldsymbol{s}_1(\theta)$ and Γ_2 by $\boldsymbol{s}_2(\theta)$.

We now have for the stream function

$$\psi(\boldsymbol{r}) = -\frac{q_1}{2\pi}\iint_{\mathscr{D}_1} \log|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}x'\,\mathrm{d}y' - \frac{q_2}{2\pi}\iint_{\mathscr{D}_2} \log|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}x'\,\mathrm{d}y'. \tag{44}$$

The angular momentum is

$$L = q_1 \iint_{\mathscr{D}_1} |\boldsymbol{r}|^2\,\mathrm{d}x\,\mathrm{d}y + q_2 \iint_{\mathscr{D}_2} |\boldsymbol{r}|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{45}$$

and the energy
$$E = q_1 \iint_{\mathscr{D}_1} \psi(\boldsymbol{r})\,\mathrm{d}x\,\mathrm{d}y + q_2 \iint_{\mathscr{D}_2} \psi(\boldsymbol{r})\,\mathrm{d}x\,\mathrm{d}y, \tag{46}$$

with ψ as given by (44).

The bounding curves are now perturbed to

$$\boldsymbol{s}_i \to \boldsymbol{s}_i + \epsilon \boldsymbol{u}_{i,1} + \tfrac{1}{2}\epsilon^2 \boldsymbol{u}_{i,2} + O(\epsilon^3). \tag{47}$$

We write for the first- and second-order area variations

$$\mathrm{d}A_{i,1}(\theta) = \boldsymbol{k}\cdot\boldsymbol{u}_{i,1}\wedge\dot{\boldsymbol{s}}_i\,\mathrm{d}\theta, \tag{48}$$

$$\mathrm{d}A_{i,2}(\theta) = \{\boldsymbol{k}\cdot\boldsymbol{u}_{i,1}\wedge\dot{\boldsymbol{u}}_i + \boldsymbol{k}\cdot\boldsymbol{u}_{i,2}\wedge\dot{\boldsymbol{s}}_i\}\,\mathrm{d}\theta. \tag{49}$$

The first index (i) denotes the curve and the second index the order of the variation.

In the special case that the $\boldsymbol{s}_i$ describe circles we have the identity

$$\boldsymbol{k}\cdot\boldsymbol{u}_{i,1}\wedge\dot{\boldsymbol{s}}_i = \boldsymbol{u}_{i,1}\cdot\boldsymbol{s}_i,$$

and we define

$$\phi_i = \boldsymbol{u}_{i,1}\cdot\boldsymbol{s}_i. \tag{50}$$

As was pointed out in §2, $\phi_i(\theta)$ can be identified with $d_i\,\delta r_i(\theta)$ where d_i is the radius of the circle and $\epsilon\delta r_i$ the $O(\epsilon)$ Lagrangian displacement. In the special case of circles we further have

$$\mathrm{d}A_{i,1}(\theta) = \phi_i(\theta)\,\mathrm{d}\theta. \tag{51}$$

With this notation we find for the first and second variations of the stream function

$$\psi_1(\boldsymbol{r}) = -\frac{q_1-q_2}{2\pi}\int_{\theta_1}^{\theta_2}\log|\boldsymbol{r}-\boldsymbol{s}_1|\,\mathrm{d}A_{1,1}(\theta) - \frac{q_2}{2\pi}\int_{\theta_1}^{\theta_2}\log|\boldsymbol{r}-\boldsymbol{s}_2|\,\mathrm{d}A_{2,1}(\theta), \tag{52}$$

$$\psi_2(\boldsymbol{r}) = -\frac{q_1-q_2}{2\pi}\left\{\int_{\theta_1}^{\theta_2}\boldsymbol{u}_{1,1}\cdot\boldsymbol{\nabla}'\log|\boldsymbol{r}-\boldsymbol{r}'|_{\boldsymbol{r}'=\boldsymbol{s}_1}\mathrm{d}A_{1,1}(\theta) + \int_{\theta_1}^{\theta_2}\log|\boldsymbol{r}-\boldsymbol{s}_1|\,\mathrm{d}A_{1,2}(\theta)\right\}$$
$$-\frac{q_2}{2\pi}\left\{\int_{\theta_1}^{\theta_2}\boldsymbol{u}_{2,1}\cdot\boldsymbol{\nabla}'\log|\boldsymbol{r}-\boldsymbol{r}'|_{\boldsymbol{r}'=\boldsymbol{s}_2}\mathrm{d}A_{2,1}(\theta) + \int_{\theta_1}^{\theta_2}\log|\boldsymbol{r}-\boldsymbol{s}_2|\,\mathrm{d}A_{2,2}(\theta)\right\}, \tag{53}$$

and for the variations of the angular momentum

$$L_1 = (q_1-q_2)\int_{\theta_1}^{\theta_2}|\boldsymbol{s}_1|^2\,\mathrm{d}A_{1,1}(\theta) + q_2\int_{\theta_1}^{\theta_2}|\boldsymbol{s}_2|^2\,\mathrm{d}A_{2,1}(\theta), \tag{54}$$

$$L_2 = (q_1-q_2)\left\{\int_{\theta_1}^{\theta_2}2\boldsymbol{u}_{1,1}\cdot\boldsymbol{s}_1\,\mathrm{d}A_{1,1}(\theta) + \int_{\theta_1}^{\theta_2}|\boldsymbol{s}_1|^2\,\mathrm{d}A_{1,2}(\theta)\right\}$$
$$+q_2\left\{\int_{\theta_1}^{\theta_2}2\boldsymbol{u}_{2,1}\cdot\boldsymbol{s}_2\,\mathrm{d}A_{2,1}(\theta) + \int_{\theta_1}^{\theta_2}|\boldsymbol{s}_2|^2\,\mathrm{d}A_{2,2}(\theta)\right\}. \tag{55}$$

3.1. *Angular momentum*

Before we investigate the energy variations, we can already establish some results by considering just the area and angular momentum functionals. Consider the variational problem with which we search for a conditional extremum in angular momentum for given areas of constant vorticity. We construct the functional

$$F = L + \lambda A(\Gamma_1) + \mu A(\Gamma_2), \tag{56}$$

where $A(\Gamma_i)$ denotes the area enclosed by curve Γ_i.

We get for the first variation of the functional F

$$F_1 = \int_{\theta_1}^{\theta_2}\{(q_1-q_2)|\boldsymbol{s}_1|^2+\lambda\}\,\mathrm{d}A_{1,1}(\theta) + \int_{\theta_1}^{\theta_2}\{q_2|\boldsymbol{s}_2|^2+\mu\}\,\mathrm{d}A_{2,1}(\theta). \tag{57}$$

Because $\mathrm{d}A_{i,1}$ is arbitrary (through the vector fields $\boldsymbol{u}_{i,1}$), we find the following two Euler equations:

$$(q_1-q_2)|\boldsymbol{s}_1|^2+\lambda = 0, \tag{58}$$

$$q_2|\boldsymbol{s}_2|^2+\mu = 0, \tag{59}$$

which show that for an extremum in angular momentum the vorticity distribution consists of a circular core of constant vorticity surrounded by a ring of the other vorticity (concentric). This clearly constitutes a stationary flow. This will be the basic vortex to be studied in this section.

Because both boundaries are circles, we can, as in the previous section, write for the second variation

$$L_2 = 2(q_1-q_2)\int_{\theta_1}^{\theta_2} \phi_1^2\,\mathrm{d}\theta + 2q_2\int_{\theta_1}^{\theta_2} \phi_2^2\,\mathrm{d}\theta, \tag{60}$$

where we have used the constraint of second-order area conservation, i.e.

$$\int \mathrm{d}A_{i,2}(\theta) = 0.$$

Definitions (50) and (51) have been invoked here. This expression proves the formal stability of a vortex with $0 \leqslant q_2 \leqslant q_1$ and $q_1 \leqslant q_2 \leqslant 0$. This is the small-amplitude form of the nonlinear stability result of Dritschel (1988).

3.2. *Energy*

Other stability regimes can be uncovered with the use of the second variation of energy. The basic state to be investigated is a circular vortex with a circular core of constant vorticity q_1 (region $\mathscr{D}_1$) surrounded by a ring of constant vorticity q_2 (region $\mathscr{D}_2$). As the bounding circles are slightly perturbed, the change in energy is

$$\Delta E = q_1 \iint_{\mathscr{D}_1+\delta\mathscr{D}_1} (\psi_0+\Delta\psi)\,\mathrm{d}x\,\mathrm{d}y + q_2 \iint_{\mathscr{D}_2+\delta\mathscr{D}_2} (\psi_0+\Delta\psi)\,\mathrm{d}x\,\mathrm{d}y - q_1\iint_{\mathscr{D}_1} \psi_0\,\mathrm{d}x\,\mathrm{d}y - q_2\iint_{\mathscr{D}_2}\psi_0\,\mathrm{d}x\,\mathrm{d}y, \tag{61}$$

where $\Delta\psi = \epsilon\psi_1 + \frac{1}{2}\epsilon^2\psi_2 + \dots$. By expanding the integrals we get

$$E_1 = (q_1-q_2)\int_{\theta_1}^{\theta_2} \psi_0(s_1)\,\mathrm{d}A_{1,1}(\theta) + q_2\int_{\theta_1}^{\theta_2}\psi_0(s_2)\,\mathrm{d}A_{2,1}(\theta) + q_1\iint_{\mathscr{D}_1}\psi_1\,\mathrm{d}x\,\mathrm{d}y + q_2\iint_{\mathscr{D}_2}\psi_1\,\mathrm{d}x\,\mathrm{d}y, \tag{62}$$

$$\begin{aligned} E_2 = {} & (q_1-q_2)\left\{\int_{\theta_1}^{\theta_2} \boldsymbol{u}_{1,1}\cdot\nabla\psi_0(s_1)\,\mathrm{d}A_{1,1}(\theta) + \int_{\theta_1}^{\theta_2}\psi_0(s_1)\,\mathrm{d}A_{1,2}(\theta)\right\} \\ & + q_2\left\{\int_{\theta_1}^{\theta_2} \boldsymbol{u}_{2,1}\cdot\nabla\psi_0(s_2)\,\mathrm{d}A_{2,1}(\theta) + \int_{\theta_1}^{\theta_2}\psi_0(s_2)\,\mathrm{d}A_{2,2}(\theta)\right\} \\ & + (q_1-q_2)\int_{\theta_1}^{\theta_2} 2\psi_1(s_1)\,\mathrm{d}A_{1,1}(\theta) + q_2\int_{\theta_1}^{\theta_2} 2\psi_1(s_2)\,\mathrm{d}A_{2,1}(\theta) \\ & + q_1\iint_{\mathscr{D}_1}\psi_2\,\mathrm{d}x\,\mathrm{d}y + q_2\iint_{\mathscr{D}_2}\psi_2\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{63}$$

By substitution of (52) in (62) and interchanging the order of integration, we find that the sum of the first two integrals in (62) is equal to the sum of the last two integrals. Since the bounding curves are streamlines, we obtain for the first variation

$$E_1 = 2(q_1-q_2)\,\psi_0(\Gamma_1)\int \mathrm{d}A_{1,1} + 2q_2\,\psi_0(\Gamma_2)\int \mathrm{d}A_{2,1}. \tag{64}$$

For area-preserving perturbations we thus have in this case too that the first variation of the energy is zero if the basic state is a stationary flow. For this we need not assume that the vortex is circular.

The second variation can be simplified by the following observations. First we note that because the second-order area change has to be zero and the stream function is constant on the bounding curves, the integrals $\int \psi_0 \, dA_{1,2}$ and $\int \psi_0 \, dA_{2,2}$ in (63) are zero. Moreover, by substitution of the expression for ψ_2 (i.e. (53)) and interchanging the order of integration, the sum of the last two integrals appearing in (63) is found to be equal to the sum of the first and the third integral. We further have

$$u_{i,1} \cdot \nabla \psi_0(r = d_i) = \frac{-v_\theta(d_i)}{d_i} \phi_i,$$

where v_θ is the tangential velocity of the circular vortex at the indicated radii and ϕ_i is defined by (50).

By using (51), we can reduce the second variation to the following form:

$$E_2 = -2(q_1 - q_2)\frac{v(d_1)}{d_1}\int_{\theta_1}^{\theta_2} \phi_1^2(\theta)\, d\theta - 2q_2 \frac{v(d_2)}{d_2}\int_{\theta_1}^{\theta_2} \phi_2^2(\theta)\, d\theta$$
$$+ 2(q_1 - q_2)\int_{\theta_1}^{\theta_2} \psi_1(s_1)\, \phi_1 \, d\theta + 2q_2 \int_{\theta_1}^{\theta_2} \psi_1(s_2)\, \phi_2 \, d\theta. \quad (65)$$

For the last two integrals appearing in (65) we find after substitution of (52) the following expressions:

$$\int_{\theta_1}^{\theta_2} \psi_1(s_1)\, \phi_1 \, d\theta = \{-(q_1 - q_2)\langle \mathcal{L}_{1,1}\phi_1, \phi_1\rangle - q_2 \langle \mathcal{L}_{2,1}\phi_2, \phi_1\rangle\}, \quad (66)$$

$$\int_{\theta_1}^{\theta_2} \psi_1(s_2)\, \phi_2 \, d\theta = \{-(q_1 - q_2)\langle \mathcal{L}_{1,2}\phi_1, \phi_2\rangle - q_2 \langle \mathcal{L}_{2,2}\phi_2, \phi_2\rangle\}, \quad (67)$$

where
$$\mathcal{L}_{i,j}\phi(\theta) = \frac{1}{\pi}\int_{\theta_1}^{\theta_2} \log|d_j e^{i\theta} - d_i e^{i\theta'}|\, \phi(\theta')\, d\theta'. \quad (68)$$

We introduced some Hilbert-space shorthand notation here. The inner product on $L^2[0, 2\pi]$ is

$$\langle f, g\rangle = \int_0^{2\pi} fg \, d\theta,$$

while the norm $\| \dots \|$ in L^2 is defined by

$$\|f\| = \langle f, f\rangle^{\frac{1}{2}}. \quad (69)$$

By writing

$$\log|d_i e^{i\theta} - d_i e^{i\theta'}| = \log|d_i| + \log|1 - e^{i(\theta' - \theta)}| = \log|d_i| + \mathcal{L},$$

with $\mathcal{L}$ the same operator as in the single vortex patch case (see (38)), we have

$$\mathcal{L}_{i,i}\phi_i(\theta) = \mathcal{L}\phi_i(\theta)$$

because
$$\log|d_i| \int_{\theta_1}^{\theta_2} \phi_i(\theta')\, d\theta' = \log|d_i| \int_{\theta_1}^{\theta_2} dA_{i,1}(\theta) = 0,$$

in view of the area conservation constraint. The operators $\mathscr{L}_{i,i}$ thus have eigenfunctions $\cos n\theta$ and $\sin n\theta$ with eigenvalues $\lambda_n = -1/n$. For $i \neq j$ we have

$$\mathscr{L}_{i,j}\,\phi_i(\theta) = \frac{1}{\pi}\int_{\theta_1}^{\theta_2} \log|1-\Lambda\,\mathrm{e}^{\mathrm{i}(\theta'-\theta)}|\,\phi_i(\theta')\,\mathrm{d}\theta',$$

where $\Lambda = d_i/d_j$. For $d_j \geqslant d_i$ the logarithm can, as in §2.3, be developed in a convergent series and it is easily seen that the eigenfunctions of this operator are the trigonometric functions with eigenvalues

$$\lambda_n = -\Lambda^n/n. \tag{70}$$

If $d_j \leqslant d_i$ one divides by d_i and proceeding as before it is found that the trigonometric functions are again the eigenfunctions with eigenvalues λ_n with $\Lambda = d_j/d_i$. Generally we have that the eigenvalues of $\mathscr{L}_{i,j}$ are

$$\lambda_n(i,j) = -\frac{\Lambda_{i,j}^n}{n} \quad (n = 1, 2, \ldots), \tag{71}$$

with

$$\Lambda_{i,j} = \min\left\{\frac{d_i}{d_j}, \frac{d_j}{d_i}\right\}, \tag{72}$$

and eigenfunctions $\cos n\theta$ and $\sin n\theta$.

Putting everything together, we have for the second variation in energy

$$E_2 = -2(q_1-q_2)\frac{v(d_1)}{d_1}\|\phi_1\|^2 - 2q_2\frac{v(d_2)}{d_2}\|\phi_2\|^2 \\ -(q_1-q_2)^2\langle\mathscr{L}_{1,1}\phi_1,\phi_1\rangle - q_2^2\langle\mathscr{L}_{2,2}\phi_2,\phi_2\rangle - 2q_2(q_1-q_2)\langle\mathscr{L}_{1,2}\phi_1,\phi_2\rangle. \tag{73}$$

Note that if we put $q_2 = 0$ the result of §2.3 is recovered as well as when we put $q_2 = q_1$.

3.3. *Generalization to a circular vortex with n rings*

Having established a working knowledge of how to calculate the different variations of energy and angular momentum for the circular vortex with two regions of different constant vorticity we can easily generalize to the case of arbitrary many nested circular regions. With an elementary calculation it can be verified that all such circular vortices provide an extremum in energy and angular momentum under area-preserving variations. The second variations of these quantities thus can again be used to establish formal stability for circular vortices with piecewise-continuous vorticity.

It is not hard to verify that for the general case of n regions with vorticities $q_i\,(i = 1, \ldots, n)$ the vortex consisting of n concentric rings with the respective vorticities q_i extremizes angular momentum for given vorticity areas. Moreover, the second variation of such a vortex is (with $i = 1$ corresponding to the circular core, and counting the rings in increasing order outwards)

$$L_2 = 2(q_1-q_2)\int_{\theta_1}^{\theta_2}\phi_1^2\,\mathrm{d}\theta + 2(q_2-q_3)\int_{\theta_1}^{\theta_2}\phi_2^2\,\mathrm{d}\theta + \ldots \\ + 2(q_{n-1}-q_n)\int_{\theta_1}^{\theta_2}\phi_{n-1}^2\,\mathrm{d}\theta + 2q_n\int_{\theta_1}^{\theta_2}\phi_n^2\,\mathrm{d}\theta. \tag{74}$$

Again, the ϕ_j are equal to $d_j\,\delta r_j(\theta)$, where $\epsilon\delta r_j$ is the $O(\epsilon)$ Lagrangian displacement of fluid elements lying on circles d_j = constant. L_2 can be cast in the following concise form:

$$L_2 = -2 \sum_{j=1}^{n} \Delta q_j \|\phi_j\|^2, \tag{75}$$

where

$$\Delta q_j = q_{j+1} - q_j. \tag{76}$$

Equation (74) proves the formal stability of a vortex with $0 < q_n < q_{n-1} < \ldots q_2 < q_1$ (i.e. all $\Delta q < 0$), or $q_1 < q_2 < \ldots < q_{n-1} < q_n < 0$ (i.e. all $\Delta q > 0$). As pointed out before, this is the small-amplitude form of the nonlinear stability result found by Dritschel (1988).

The second variation of the energy for the general case of n vorticities is

$$E_2 = -\sum_{i=1}^{n} 2\Delta q_i \int \boldsymbol{u}_{i,1} \cdot \nabla \psi_0(\boldsymbol{s}_i)\, \phi_i(\theta)\, \mathrm{d}\theta - \sum_{i=1}^{n} 2\Delta q_i \int \psi_1(\boldsymbol{s}_i)\, \phi_i(\theta)\, \mathrm{d}\theta, \tag{77}$$

where

$$\boldsymbol{u}_{i,1} \cdot \nabla \psi_0(\boldsymbol{s}_i) = -\frac{v_\theta(d_i)}{d_i} \phi_i(\theta), \quad v_\theta(r) = \frac{1}{r} \int^r s q(s)\, \mathrm{d}s,$$

and

$$\psi_0(\boldsymbol{r}) = -\sum_{i=1}^{n} \frac{q_i}{2\pi} \iint_{\mathscr{D}_i} \log |\boldsymbol{r} - \boldsymbol{r}'|\, \mathrm{d}x'\, \mathrm{d}y', \tag{78}$$

$$\psi_1(\boldsymbol{s}_i) = \sum_{j=1}^{n} \frac{\Delta q_j}{2\pi} \int \log |\boldsymbol{s}_i - \boldsymbol{s}_j(\theta')|\, \phi_j(\theta')\, \mathrm{d}\theta'. \tag{79}$$

By substitution we find the following expression:

$$E_2 = \sum_{i=1}^{n} \frac{2 v_\theta(d_i)}{d_i} \Delta q_i \|\phi_i\|^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta q_i\, \Delta q_j \langle \mathscr{L}_{i,j} \phi_i, \phi_j \rangle, \tag{80}$$

where $\mathscr{L}_{i,j}$ is the operator defined by (68).

4. Linear and formal stability of vortices

With the expressions for the second variations of energy and angular momentum, linear and formal stability regimes can be uncovered in the following manner. It can be shown that the linearized dynamics conserves E_2 and L_2 (see Appendix C, or for a generalization Holm *et al.* 1985). Consider the evolution of a single decaying or growing normal mode. Since both E_2 and L_2 are proportional to the normal mode's amplitude squared, they must both vanish identically for such modes. Thus a sufficient condition for linear stability is that for all perturbations with $L_2 = 0$, E_2 is sign definite, or vice versa.

Stability to particular wavenumber perturbations can be deduced by restricting the analysis to a single harmonic component and replacing the norm of $\mathscr{L}_{i,j}$ by $A_{i,j}/m$ with m the particular wavenumber. In general, the mathematical structure of the problem is such that we have to determine the sign of a quadratic form $(\delta\boldsymbol{r}, \boldsymbol{E}\delta\boldsymbol{r})$ along a manifold $(\delta\boldsymbol{r}, \boldsymbol{L}\delta\boldsymbol{r}) = 0$, where $\delta\boldsymbol{r}$ is an n-dimensional vector and $\boldsymbol{E}$ and $\boldsymbol{L}$ are $n \times n$ matrices.

Formal stability is investigated by considering the quadratic form $E_2 + \mu L_2$, where μ is an arbitrary constant. Formal stability follows if a μ can be found such that this form is sign definite. As we show below the procedure amounts to determining eigenvalues $\lambda(\mu)$ of $n \times n$ matrices. Formal stability is proven if there is a μ such that all λ are either positive or negative.

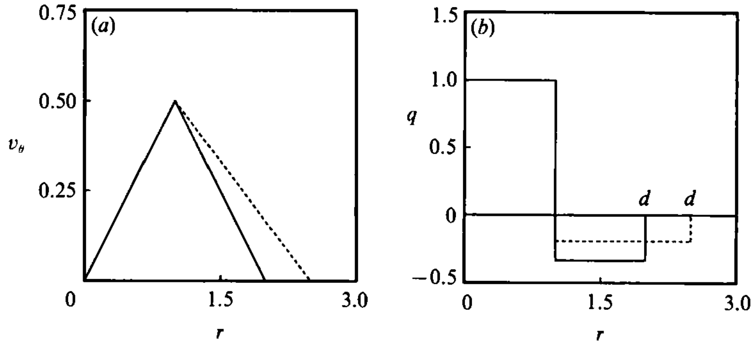

FIGURE 2. Diagram of the isolated model vortex investigated by Flierl (1988). In (*a*) the velocity profile is shown and in (*b*) the corresponding vorticity. Non-dimensionally the profiles have maximum velocity at $r = 1$ and zero velocity beyond $r = d$.

4.1. *Example: linear stability of an isolated vortex with* $n = 2$

We will use the second variations of energy and angular momentum to establish the linear stability regimes of an isolated vortex consisting of two concentric rings of oppositely signed vorticity, i.e. a vortex with $q_1 > 0$ and $q_2 < 0$ such that $v_\theta(d_2) = 0$. This vortex has zero net integrated vorticity (vanishing circulation) and serves as a simple model for more general isolated vortices which satisfy Rayleigh's inflexion-point criterion (Drazin & Reid 1981), i.e. satisfy the necessary condition for instability, and are therefore possibly unstable. This is the isolated model vortex depicted in figure 2 the linear stability of which was investigated with normal-modes analysis by Flierl (1988).

For the given basic state we have

$$v_\theta(r) = \frac{1}{r}\int^r sq(s)\,\mathrm{d}s = \tfrac{1}{2}q_1 r \qquad (0 \leqslant r \leqslant d_1), \tag{81}$$

$$= \frac{1}{2}\frac{q_1 d_1^2}{r} + \tfrac{1}{2}q_2\frac{r^2 - d_1^2}{r} \qquad (d_1 \leqslant r \leqslant d_2). \tag{82}$$

The isolated vortex has vanishing azimuthal velocity on the outer boundary, and with (82) this implies

$$q_1 \Lambda^2 + q_2(1 - \Lambda^2) = 0, \tag{83}$$

with $\Lambda = d_1/d_2$. Without loss of generality we can take $q_1 = 1$ and $d_1 = 1$ and we will throughout this section use the following relation:

$$q = -q_2 = \frac{1}{d^2 - 1},$$

or

$$d = d_2 = \left(\frac{q+1}{q}\right)^{\frac{1}{2}}, \tag{84}$$

which follow by using (83). By substitution of these relations in (75) and (80) we have for L_2 and E_2

$$L_2 = (1+q)\,\|\phi_1\|^2 - q\|\phi_2\|^2, \tag{85}$$

$$E_2 = -(1+q)\,\|\phi_1\|^2 - (1+q)^2\langle \mathscr{L}_{1,1}\phi_1, \phi_1\rangle - q^2\langle \mathscr{L}_{2,2}\phi_2, \phi_2\rangle - 2q(1+q)\langle \mathscr{L}_{1,2}\phi_1, \phi_2\rangle.$$

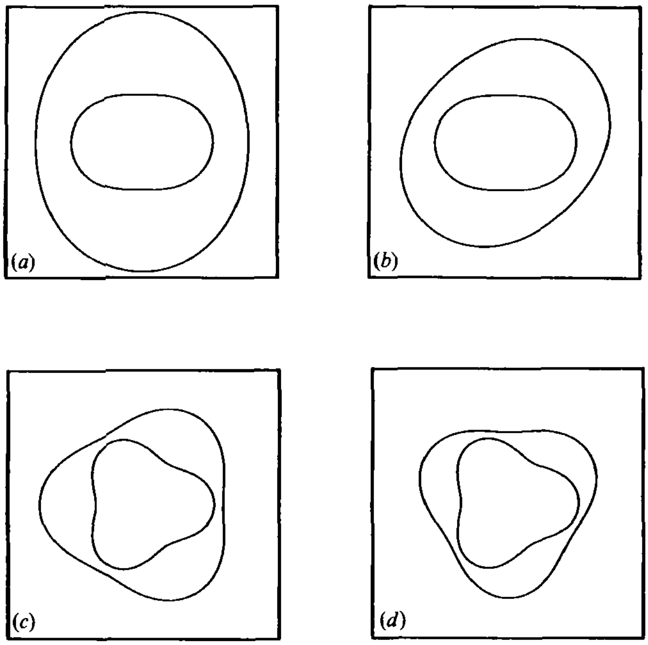

FIGURE 3. Graphs showing the structure of the possibly unstable modes of the isolated vortex model of figure 2 as determined by a consideration of the second-order variations of energy and angular momentum. In (a) and (b) the structure is shown for wavenumber-2 perturbations for (a) $q \approx \frac{1}{3}$ and (b) $q = \frac{1}{2}$. In (c) and (d) the structure is shown for wavenumber-3 perturbations with $q = 1/\sqrt{2}$ in (c) and $q = 1$ in (d). Further details are given in the text.

We will now consider the sign of these quantities with respect to a given single harmonic perturbation.

As we saw in §3, if the two bounding curves s_1 and s_2 are circles of radii d_1 and d_2, respectively, then

$$\phi_i = \boldsymbol{u}_i \cdot \boldsymbol{s}_i(\theta) = d_i\, \delta r_i(\theta),$$

with $\epsilon \delta r_i(t)$ the $O(\epsilon)$ Lagrangian displacement of the boundaries. We introduce the Fourier transforms for the perturbed boundaries:

$$\phi_i = d_i\, \delta r_i(\theta) = \sum_{k=1}^{\infty} a_{i,k} \cos k\theta + b_{i,k} \sin k\theta. \tag{86}$$

If a perturbation consists of a single harmonic component, then we take

$$\phi_1(\theta) = \mathrm{d}_1\, \delta r_1(\theta) = r_1 \cos m\theta, \tag{87}$$

$$\begin{aligned}\phi_2(\theta) &= d_2\, \delta r_2(\theta) = r_2 \cos(m\theta + m\theta_0),\\ &= r_2\{\cos m\theta_0 \cos m\theta - \sin m\theta_0 \sin m\theta\},\end{aligned} \tag{88}$$

where r_1, r_2 are positive constants. The term $m\theta_0$ represents the phase difference between the perturbations on the inner and outer circle. In figure 3 examples are shown of such phase shifts. So, the Fourier components are in this case

$$a_{1,m} = r_1, \quad b_{1,m} = 0,$$
$$a_{2,m} = r_2 \cos m\theta_0, \quad b_{2,m} = -r_2 \sin m\theta_0$$

(with all $a_{i,n}$ and $b_{i,n}$ zero for $n \neq m$). For all the norms we have

$$\|\phi_i\|^2 = \pi r_i^2.$$

With these preliminaries we now write for the second variation of the energy

$$E_{2,m} = -(1+q)\,\|\phi_1\|^2 + (1+q)^2\frac{\|\phi_1\|^2}{m} + q^2\frac{\|\phi_2\|^2}{m} - 2q(1+q)\frac{\Lambda^m}{m}\,\|\phi_1\|\,\|\phi_2\|\cos m\theta_0, \tag{89}$$

where, with (84),
$$\Lambda = \frac{1}{d} = \left(\frac{q}{q+1}\right)^{\frac{1}{2}}.$$

We have added an index m to E_2 to emphasize that this expression gives the second-order change in energy due to a single harmonic perturbation with wavenumber m.

Let us first consider the case of $m = 1$. A single $m = 1$ perturbation merely shifts its position. For two concentric circles simultaneous $m = 1$ perturbations can shift the inner circle to a position eccentric to the enclosing outer circle unless the phase difference is zero (i.e. for $\theta_0 = 0$). We need to investigate the sign of E_2 along a 'direction' in which $L_2 = 0$. Using (85) to eliminate the occurrence of $\|\phi_1\|$ in E_2, we have for $m = 1$

$$E_{2,m=1} = 2q^2\|\phi_2\|^2\{1-\cos\theta_0\}.$$

We see that $E_{2,m=1} = 0$ whenever the phase difference is zero. This was to be expected since in this case the vortex is merely displaced and this certainly does not change the energy. Whenever the phase difference is not zero, the second variation is positive. So, modulo translations of the entire vortex, it can be concluded that the isolated vortex is linearly stable with respect to $m = 1$ perturbations. This is in agreement with the observation of Stern (1987) that a configuration with one of the circles displaced with respect to the other constitutes a propagating vortex which in a co-moving frame is stationary (this is sometimes viewed as being an unstable degenerate mode with a linear growth rate; see Flierl 1988).

For the $m = 2$ mode we get

$$E_{2,m=2} = q\|\phi_2\|^2\left\{-\tfrac{1}{2}+q-q\left(\frac{q}{1+q}\right)^{\frac{1}{2}}\cos 2\theta_0\right\}. \tag{90}$$

In contrast to the case of $m = 1$, E_2 now depends on both q and the phase difference. As the phase difference is varied, $E_{2,m=2}$ varies between

$$-\tfrac{1}{2}+q-q\left(\frac{q}{1+q}\right)^{\frac{1}{2}} \leqslant \frac{E_{2,m=2}}{(q\|\phi_2|^2)} \leqslant -\tfrac{1}{2}+q+q\left(\frac{q}{1+q}\right)^{\frac{1}{2}}.$$

If the upper and lower bound are of the same sign, then the vortex is linearly stable to wavenumber-2 perturbations. Note that q can be given any positive value. For large q-values the lower bound tends to $-\frac{1}{2}$ whereas the upper bound tends to $2q$. So $E_{2,m=2}$ is possibly of fixed sign only for small q. We see that for very small q the upper and lower bounds are both negative. The value of q for which the upper bound becomes zero marks a critical value below which the vortex is linearly stable with respect to wavenumber-2 perturbations. To determine this critical value we solve for q by equating the upper bound to zero, to find, with a few simple algebraic manipulations, that $q_{\text{crit}} = \frac{1}{3}$. So, if $q < q_{\text{crit}}$ (by substitution $d > 2$), the vortex is linearly stable. This critical value was previously found by Flierl (1988) by means of a normal-modes analysis of the linearized equations of motion. He showed that for $q > q_{\text{crit}}$ the normal modes grow exponentially in time whereas for $q < q_{\text{crit}}$ they are neutrally stable. By a consideration of the variations of the conserved quantities of the system, we thus find the same stable regime. The stable regime can be interpreted as being such that all small $m = 2$ perturbations lead to an increase in energy.

With (90) we see that the phase difference for a possible unstable mode is determined by the relation

$$\cos 2\theta_0 = \frac{(q-\frac{1}{2})(1+q)^{\frac{1}{2}}}{qq^{\frac{1}{2}}}. \tag{91}$$

This shows that for values very close to the critical value $q = \frac{1}{3}$ the phase difference approaches $\frac{1}{2}\pi$ whereas for instance for $q = \frac{1}{2}$ the phase difference of the unstable mode is $\frac{3}{4}\pi$ (see figure 3*a*, *b*). For very large q the phase difference approaches $\theta_0 = \pi$. Using (85) we find that

$$\frac{\delta r_2}{\delta r_1} = \frac{1}{d}\left(\frac{1+q}{q}\right)^{\frac{1}{2}} = 1.$$

As a final example consider the case of $m = 3$. For $E_{2, m=3}$ we find

$$E_{2, m=3} = \tfrac{2}{3}q\|\phi_2\|^2\left\{-1+q-\frac{q^2}{1+q}\cos 3\theta_0\right\}.$$

So,
$$-1+q-\frac{q^2}{1+q} \leqslant \frac{\frac{3}{2}E_{2, m=3}}{(q\|\phi_2\|^2)} \leqslant -1+q+\frac{q^2}{1+q}.$$

As in the previous example, only for sufficiently small q is E_2 single-signed (negative definite). The critical value is easily determined to be $q_{\text{crit}} = 1/\surd 2$. Therefore, for $q < 1/\surd 2 (d > (1+\surd 2)^{\frac{1}{2}})$ the isolated vortex is linearly stable. This, again, is in agreement with Flierl's results. The phase difference for the unstable modes with $q > 1/\surd 2$ is determined by the following equation:

$$\cos 3\theta_0 = (q^2-1)/q^2.$$

So, near the critical value the phase difference is close to $\theta_0 = \frac{1}{3}\pi$, whereas for $q = 1$ it is $\theta_0 = \frac{1}{2}\pi$ (see figure 3*c*, *d*). For very large q the phase difference gets close to $\frac{2}{3}\pi$.

In a similar vein the case of $q_2 > q_1 > 0$ can be investigated. An inspection of the signs of E_2 and L_2 leads to the discovery of the same stability boundaries as found by Flierl (1988) with his normal-modes analysis of the linearized equations of motion. Also, the linear stability of the annular vortex, i.e. the case $q_1 = 0, q_2 \neq 0$, is easily investigated with the above-derived expressions for the second variations. The stability regimes previously discovered by Michalke & Timme (1967) are again found. So, remarkably, with the expressions for the second variations we can find not only the linear stability regimes but also the structure of unstable normal modes. Growth or decay rates are not derivable from these considerations.

4.2. *Example: formal stability of an isolated vortex with* $n = 2$

In order to investigate formal stability, which is stronger than linear stability (Holm *et al.* 1985), we consider the quadratic form $E_2 + \frac{1}{2}\mu L_2$, where μ is an arbitrary scalar (we have added a factor $\frac{1}{2}$ for convenience). In vector notation, this quadratic form can be written as $(\delta \boldsymbol{r}, (\boldsymbol{E}+\frac{1}{2}\mu\boldsymbol{L})\,\delta\boldsymbol{r})$. If we can find a μ for which all of the eigenvalues, $\lambda(\mu)$, of the matrix $\boldsymbol{E}+\frac{1}{2}\mu\boldsymbol{L}$ are of the same sign then formal stability is proven.

It will be convenient to use the spectral form of the second variations, which are, with (86),

$$L_2 = -2\pi \sum_{i=1}^{n} \Delta q_i \left(\sum_{m=1}^{\infty} a_{i,m}^2 + b_{i,m}^2 \right), \tag{92}$$

$$E_2 = \pi \sum_{i=1}^{n} 2 \frac{v_\theta(d_i)}{d_i} \Delta q_i \left(\sum_{m=1}^{\infty} a_{i,m}^2 + b_{i,m}^2 \right) + \pi \sum_{i=1}^{n} \Delta q_i^2 \left(\sum_{m=1}^{\infty} \frac{a_{i,m}^2 + b_{i,m}^2}{m} \right)$$
$$+ 2\pi \sum_{i=1}^{j-1} \sum_{j=2}^{n} \Delta q_i \Delta q_j \sum_{m=1}^{\infty} \frac{A_{i,j}^m}{m} (a_{i,m} a_{j,m} + b_{i,m} + b_{j,m}). \tag{93}$$

As an example we consider the special case of the previous section again. We write

$$a_{i,m} = r_{i,m} \cos\theta_{i,m}, \quad b_{i,m} = -r_{i,m} \sin\theta_{i,m} \quad (i = 1, 2),$$

from which it follows that $a_{i,m}^2 + b_{i,m}^2 = r_{i,m}^2$ and

$$a_{1,m} a_{2,m} + b_{1,m} b_{2,m} = r_{1,m} r_{2,m} \cos(\theta_{2,m} - \theta_{1,m}) = \epsilon_m r_{1,m} r_{2,m},$$

where ϵ_m is some number between -1 and $+1$. We now form the quadratic functional $E_2 + \frac{1}{2}\mu L_2$ which in spectral form is

$$E_2 + \tfrac{1}{2}\mu L_2 = \pi \sum_{m=1}^{\infty} (1+q) \left\{ \mu - 1 + \frac{1+q}{m} \right\} r_{1,m}^2 + \pi \sum_{m=1}^{\infty} q \left(\frac{q}{m} - \mu \right) r_{2,m}^2$$
$$+ 2\pi \sum_{m=1}^{\infty} \frac{q(1+q)}{d^m m} \epsilon_m r_{1,m} r_{2,m}. \tag{94}$$

Basically this is a sum of quadratic forms each of which is proportional to

$$\alpha_m x^2 + 2\gamma_m xy + \beta_m y^2,$$

with

$$\alpha_m = (1+q)\left(\mu - \frac{m-1}{m} + \frac{q}{m} \right), \quad \beta_m = q\left(\frac{q}{m} - \mu \right), \quad \gamma_m = \epsilon_m \frac{q^{1+\frac{1}{2}m}(1+q)^{1-\frac{1}{2}m}}{m}, \tag{95}$$

where to calculate γ_m we have used (84). Formal stability follows if there is a μ such that for all m the eigenvalues of the matrix

$$\begin{pmatrix} \alpha_m & \gamma_m \\ \gamma_m & \beta_m \end{pmatrix}$$

are both positive or both negative. This is impossible if we include the $m = 1$ component because $\alpha_1 = (1+q)(\mu - q)$ and $\beta_1 = q(q-\mu)$. Thus no matter what μ is taken to be, these coefficients differ in sign and the quadratic form is not sign-definite. As discussed above, a wavenumber-1 perturbation leads to a steady translational motion of the isolated vortex, i.e. such a perturbation imparts impulse to the vortex. The $m = 1$ modes are the only modes that have non-vanishing impulse. They are entirely uncoupled from the higher wavenumber modes in the nonlinear as well as in the linear dynamics. This follows from the conservation of impulse; if the initial structure has zero impulse (i.e. no $m = 1$ component) then none can be generated. If we exclude this component we find that for small enough q there is always a μ which makes the quadratic forms with $m \geq 2$ negative definite. This is shown as follows.

The eigenvalues are

$$\lambda_m = \tfrac{1}{2}(\alpha_m + \beta_m) \pm \tfrac{1}{2}((\alpha_m - \beta_m)^2 + 4\gamma_m^2)^{\frac{1}{2}},$$

which we write as $\lambda_m = \frac{1}{2}(\alpha_m+\beta_m) \pm \frac{1}{2}((\alpha_m+\beta_m)^2+\mathscr{R}_m)^{\frac{1}{2}}$,

where $\mathscr{R}_m = (\alpha_m-\beta_m)^2+4\gamma_m^2-(\alpha_m+\beta_m)^2.$

Both eigenvalues are clearly negative if we can show that

$$\alpha_m+\beta_m < 0, \quad \mathscr{R}_m < 0, \tag{96a, b}$$

Substitution of the above expressions for α_m, β_m, and γ_m shows that for $m = 2$ we need simultaneously

$$q^2+\mu-\tfrac{1}{2} < 0, \quad \mu^2-\tfrac{1}{2}\mu+\frac{q}{4(1+q)} < 0. \tag{97a, b}$$

From (97b) we infer that $\mu > 0$ and then from (97a) it follows that

$$0 < \mu < \tfrac{1}{2}-q^2. \tag{98}$$

Formal stability will therefore only be possible for $q < 1/\sqrt{2}$. More generally condition (96a) implies

$$0 < \mu < \frac{m-1}{m}+\frac{m-2}{m}q-\frac{2}{m}q^2.$$

It is easily verified that for larger m this becomes less and less restrictive on μ so if (98) is satisfied then automatically (96a) is satisfied for all $m \geqslant 2$.

The question now becomes whether in the range given by (98) we can find a μ such that for all $m \geqslant 2$ condition (96b) is also satisfied. Substitution provides us with the following expression:

$$\frac{\mathscr{R}_m}{4q(1+q)} = f_{m,q}(\mu) = \mu^2-\frac{m-1}{m}\mu+\left(\frac{m-2}{m^2}\frac{q^2}{1+q}+\frac{m-1}{m^2}\frac{q}{1+q}+\frac{q^{1+m}(1+q)^{2-m}-q^3}{m^2(1+q)}\right). \tag{99}$$

The function $f_{m,q}$ is convex in μ with a minimum at

$$\mu_{\min} = (m-1)/2m.$$

In particular for $m = 2$ this minimum is

$$f_{2,q}(\mu_{\min}) = \frac{-1}{16}+\frac{q}{4(1+q)}$$

which is zero for $q = \frac{1}{3}$ and negative only for $q < \frac{1}{3}$. For $q < \frac{1}{3}$, $f_{m,q}$ is negative in the range

$$\frac{1}{4}-\frac{1}{4}\left(\frac{1-3q}{1+q}\right)^{\frac{1}{2}} < \mu < \frac{1}{4}+\frac{1}{4}\left(\frac{1-3q}{1+q}\right)^{\frac{1}{2}},$$

which is within the range allowed by (98).

For large m we infer from (99) that the interval for negative $f_{m,q}$ tends to $0 < \mu < 1$. More detailed inspection of (99) shows that the interval for negative $f_{m+1,q}$ overlaps that of $f_{m,q}$ for any $m \geqslant 2$. We conclude therefore that for any $q < \frac{1}{3}$ there is always a μ such that the quadratic form $E_2+\frac{1}{2}\mu L_2$ is negative definite if we exclude perturbations that set the vortex into translational motion (wavenumber-1 perturbations). We also conclude that the range of q-values for which the vortex is formally stable coincides with the linear stability range ($0 < q < \frac{1}{3}$).

Clearly the method employed here can be extended to different model vortices with more regions of different vorticity but typically a computer will be needed to search for positive- or negative-definite eigenvalues of the quadratic forms as a function of the multiplier μ.

5. Discussion

We have derived new expressions for the second variations of energy and angular momentum under area-preserving perturbations for circular vortices with piecewise-constant vorticity. With these expressions we can investigate the formal stability of vortices with piecewise-constant vorticity. These vortices are models for isolated vortices with a continuous vorticity distribution, which cannot be proven stable by Arnol'd's method (1966) (see Appendix A). As an example we have treated the isolated-vortex model of Flierl (1988) and we have shown that the linear stability regime coincides with the formal stability range. This stability excludes the possibility of perturbations which initially change the impulse of the vortex. The fact that there are formally stable isolated vortices indicates that it might be possible to prove nonlinear stability in certain cases. But, the second variations used to prove formal stability are only the quadratic terms of the fully nonlinear expressions. The full angular momentum is of fourth order in variations of material displacements. The energy, if developed in a Taylor series expansion, contains contributions at all orders. Therefore angular momentum can constrain the energy variations only up to fourth order. At higher order the energy variations are unconstrained and usually not sign-definite. This suggests that the only form of stability that will possibly follow is conditional, that is, stability for finite but small enough perturbations only.

This is supported by the results of Flierl (1988). He finds for the model that we have treated that around the stability limit (i.e. $q_{\text{crit}} = \frac{1}{3}$), the instabilities are subcritical. This means here that in the unstable region the instability does not saturate in a nearby (stable) state. Usually in a standard (one-parameter) subcritical bifurcation, one has that on the stable side of the bifurcation point there are nearby unstable branches. The stable branch has a stable domain around it, and as long as the perturbations are not too large, the system will remain close to the stable state.

Arnol'd nonlinear stability on the other hand is very strong; it is the analogue of the stability of a particle in an infinitely deep convex potential well. The analogue for stable isolated vortices must be the stability of a finite-depth potential well. We intend to explore this topic further in the future.

The results can further be generalized to the case of vortices with continuous vorticity by considering an ever finer partition of vorticity rings and taking the limit $n \rightarrow \infty$. Let us first investigate what this implies for the second variation of angular momentum. According to (75) we have

$$\left.\frac{\mathrm{d}^2 L}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} = L_2 = -\sum_{i=1}^{n} 2\Delta q_i\, d_i^2 \|\delta r_i\|^2. \tag{100}$$

Remember that $\epsilon\delta r_i(\theta)$ is the radial displacement of fluid elements that lie initially on the circle $r = d_i =$ constant. We can also write for the second variation

$$L_2 = -2\sum_{i=1}^{n} d_i \frac{\Delta q_i}{\Delta d_i} \|\delta r_i\|^2 d_i\, \Delta d_i,$$

with $\Delta d_i = d_i - d_{i-1}$. By taking ever smaller partitions and letting $n \rightarrow \infty$ we thus obtain the following expression:

$$L_2 = -2\iint_0^{2\pi} \left(r\frac{\partial q}{\partial r}\right) \delta r^2(r,\theta)\, r\, \mathrm{d}\theta\, \mathrm{d}r. \tag{101}$$

The integration extends over all of the vortex. Wherever jump discontinuities occur in the vorticity distribution, locally one replaces the gradient in this expression by the jump value. The boundary of the vortex can be at infinity or be at finite radius without the vorticity necessarily being continuous there. The quantity $\delta r(r,\theta)$ is the $O(\epsilon)$ displacement of a material fluid element on the circle $r=$ constant as a function of its azimuthal coordinate on the circle. It is seen that the sign of L_2 is definite whenever the sign of the vorticity gradient is of fixed sign. We thus have established the formal stability of a vortex for which $\partial q/\partial r$ is of fixed sign for all r. Moreover, the formal stability points out the possibility that the vortex can be proved to be nonlinearly stable, and this has indeed recently been accomplished (Dritschel 1988; Carnevale & Shepherd 1990).

For the first part of the second variation of the energy as given by (80) we write

$$\sum_{i=1}^{n} \frac{2v_\theta(d_i)}{d_i}\frac{\Delta q_i}{\Delta d_i} d_i \|\delta r_i\|^2 d_i\,\Delta d_i.$$

In the limit $n\rightarrow\infty$ this becomes

$$\iint_0^{2\pi} 2v(r)\frac{\partial q}{\partial r}\delta r^2(r,\theta)\,r\,\mathrm{d}\theta\,\mathrm{d}r.$$

The magnitude of the second part can be bounded from above as follows. First we note that the norm of the operators $\mathscr{L}_{i,j}$ is $\max\{\|\mathscr{L}_{i,j}\|\}=\Lambda_{i,j}$ (if we exclude wavenumber-1 perturbations this has to be replaced by $\frac{1}{2}\Lambda_{i,j}$). So,

$$|\langle \mathscr{L}_{i,j}\phi_j,\phi_i\rangle| \leqslant \Lambda_{i,j}\|\phi_i\|\,\|\phi_j\|.$$

This is a sharp estimate, i.e. there are situations in which the upper bound is actually attained. By introducing the notation

$$\phi_i' = \Delta q_i\,\phi_i$$

we have

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta q_i\Delta q_j\langle\mathscr{L}_{i,j}\phi_j,\phi_i\rangle\right| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|\langle\mathscr{L}_{i,j}\phi_j',\phi_i'\rangle| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{i,j}\|\phi_i'\|\,\|\phi_j'\|. \quad (102)$$

This too is a sharp estimate. The upper bound can in turn be cast in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{i,j}\|\phi_i'\|\,\|\phi_j'\| = 2\sum_{i\leqslant j}\frac{d_i}{d_j}\|\phi_i'\|\,\|\phi_j'\| - \sum_{i=1}^{n}\|\phi_i'\|^2 \leqslant 2\sum_{i=1}^{j}\sum_{j=1}^{n}\frac{d_i}{d_j}\|\phi_i'\|\,\|\phi_j'\|, \quad (103)$$

where $d_j\geqslant d_i$ for all $j\geqslant i$. By substitution we find that this term is equal to

$$2\sum_{i=1}^{j}\sum_{j=1}^{n}\frac{d_i}{d_j}d_i d_j|\Delta q_i|\,|\Delta q_j|\,\|\delta r_i\|\,\|\delta r_j\| = 2\sum_{i=1}^{j}\sum_{j=1}^{n}\left|\frac{\Delta q_i}{\Delta d_i}\right|\left|\frac{\Delta q_j}{\Delta d_j}\right|\|\delta r_i\|\,\|\delta r_j\|\,d_i^2\,\Delta d_i\,\Delta d_j.$$

By taking the limit this becomes equal to

$$2\int_0^\infty\left|\frac{\partial q}{\partial r}\right|\|\delta r\|(r)\left\{\int_0^r\left|\frac{\partial q}{\partial r}\right|(s)\,\|\delta r\|(s)\,s^2\,\mathrm{d}s\right\}\mathrm{d}r. \quad (104)$$

This establishes the following upper and lower bounds for the second variation of the energy:

$$2\int_0^\infty rv(r)\frac{\partial q}{\partial r}\|\delta r\|^2(r) - \left|\frac{\partial q}{\partial r}\right|\|\delta r\|(r)\left\{\int_0^r\left|\frac{\partial q}{\partial r}\right|(s)\,\|\delta r\|(s)\,s^2\,\mathrm{d}s\right\}\mathrm{d}r \leqslant E_2$$

$$\leqslant 2\int_0^\infty rv(r)\frac{\partial q}{\partial r}\|\delta r\|^2(r) + \left|\frac{\partial q}{\partial r}\right|\|\delta r\|(r)\left\{\int_0^r\left|\frac{\partial q}{\partial r}\right|(s)\,\|\delta r\|(s)\,s^2\,\mathrm{d}s\right\}\mathrm{d}r. \quad (105)$$

Using the sharp upper bound E_2^+, we can prove formal stability if there is a μ such that the quadratic form $E_2^+ + \mu L_2$ is negative definite or, vice versa, by using the sharp lower bound E_2^-, if the form $E_2^- + \mu L_2$ is positive definite. For given vortex structure the properties of the integral operators appearing in the expressions for L_2 and E_2 need to be investigated. This is left for future work.

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Appendix A. Arnol'd's method for isolated vortices

Arnol'd's (1966) method is based on the observation that energy

$$E = \frac{1}{2}\int (\nabla\psi)^2 \,\mathrm{d}x\,\mathrm{d}y$$

and generalized enstrophy $Q_F = \int F(q)\,\mathrm{d}x\,\mathrm{d}y$ (for arbitrary F) are conserved quantities for ideal planar flow. Here ψ is the stream function, q vorticity, and F an arbitrary function of one variable. For a stationary flow with stream function Ψ and vorticity Q there is a functional relation between vorticity and stream function $\Psi = \Psi(Q) = G(Q)$ where G is some (possibly multivalued) function. We consider ψ as the basic independent field, with q determined diagnostically from ψ according to $q = -\nabla^2\psi$. An arbitrary non-stationary flow $\psi(x, y; t)$ is written as the sum of the stationary state Ψ and a perturbation $\delta\psi(x, y; t)$. Now the quantity $H[\Psi+\delta\psi]-H[\Psi]$ with $H = E+Q_F$ is clearly conserved. If $F(q)$ is chosen such that

$$\Psi(Q) = G(Q) = -F'(Q)$$

then by partial integration we have

$$H[\Psi+\delta\psi]-H[\Psi] = \text{constant} = \int \tfrac{1}{2}(\nabla\delta\psi)^2 + F(Q+\delta q) - F(Q) - F'(Q)\,\delta q \,\mathrm{d}x\,\mathrm{d}y.$$

In this expression the first part is the perturbative kinetic energy. The last two terms are clearly the first two terms of the Taylor series expansion of $F(q)$ around $q = Q$. If the second derivative of F is bounded for all Q, i.e. $-\infty < c \leqslant F''(Q) \leqslant C < \infty$, then by the mean value theorem, the following estimate results:

$$\frac{1}{2}\int (\nabla\delta\psi)^2 + c(\delta q)^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant H[\Psi+\delta\psi]-H[\Psi] \leqslant \frac{1}{2}\int (\nabla\delta\psi)^2 + C(\delta q)^2 \,\mathrm{d}x\,\mathrm{d}y. \quad \text{(A 1)}$$

If the lower bound c is positive, a norm on the perturbation can be defined by

$$\|\delta\psi\| = \left\{\int (\nabla\delta\psi)^2 + c(\nabla^2\delta\psi)^2 \,\mathrm{d}x\,\mathrm{d}y\right\}^{\frac{1}{2}}, \quad \text{(A 2)}$$

and with (A 1) it then follows that

$$\|\delta\psi\|(t) \leqslant (C/c)^{\frac{1}{2}}\,\|\delta\psi\|(0). \quad \text{(A 3)}$$

The norm of the perturbative field can therefore be made arbitrarily small by choosing the initial perturbations small enough. This is a case of nonlinear Liapunov stability and is valid for perturbations of arbitrary amplitude (Carnevale & Frederiksen 1987; McIntyre & Shepherd 1987). Note that $c = \min\{-\Psi'(Q)\}$ and $C = \max\{-\Psi'(Q)\}$. An expression similar to (A 3) with a norm based entirely on disturbance enstrophy can be obtained for either zonally symmetric flows (see Shepherd 1987) or circularly symmetric flows (see Carnevale & Shepherd 1990) using linear momentum or angular momentum, respectively.

On the other hand, if C is negative, then Arnol'd showed that normed stability follows if the integral

$$\int (\nabla\delta\psi)^2 + C(\delta q)^2 \,\mathrm{d}x\,\mathrm{d}y \tag{A 4}$$

is negative definite.

For a circular vortex the tangential velocity is $v = -\partial_r \psi$ and for a stationary circularly symmetric vortex one has

$$-\Psi'(Q) = \frac{-\mathrm{d}\Psi}{\mathrm{d}r}\bigg/\frac{\mathrm{d}Q}{\mathrm{d}r} = V(r)/Q'(r), \tag{A 5}$$

where V is the tangential velocity of the stationary vortex. Consider the case of an isolated vortex in which V is positive everywhere and $Q'(r)$ changes sign only once. It is impossible to prove Arnol'd stability using directly (A 5) because $c < 0$ and $C > 0$.

If we include the angular momentum, which is also a conserved quantity, in the above expression for $H[\Psi]$, then the right-hand side of (A 5) is replaced by

$$D(r) = \frac{(V(r) + \Omega r)}{Q'(r)},$$

where Ω is an arbitrary Lagrange multiplier. If the vorticity distribution has two or more inflexion points, no Ω can be found that makes $D(r)$ positive or negative everywhere and Arnol'd stability cannot be proven. However, if a profile has exactly one inflexion point at $r = r_0$ then the choice

$$\Omega = -V(r_0)/r_0$$

may yield a $D(r)$ which is sign definite. It is easily verified that the families of profiles listed in table 1 for instance have negative definite $D(r)$ (they correspond to streamfunctions $\Psi(r) = (1+r^2)^{-\alpha}$ ($\alpha = \frac{1}{2}, 1, 2$) and $\Psi(r) = \exp(-\frac{1}{2}r^2)$, respectively) with this choice of Ω. In fact all isolated vortices with one inflexion point have negative $D(r)$. It may also be noted here that the function $\Psi(Q)$ has two branches, i.e. is multivalued for vortices with an inflexion point.

Thus only Arnol'd's second criterion which applies when C is negative is appropriate for isolated vortices. However, one can show that, for instance, all of the isolated vortices of table 1 fail to satisfy this criterion in an unrestricted domain. If we consider the integral in equation (A 4), we note that the second term, which is negative, can always be made to dominate the integral if the scale of the perturbation is made sufficiently small, and the first term, which is positive, will dominate if the scale of the perturbations is sufficiently large. Thus the only way to guarantee that this integral is negative for all perturbations is to restrict the size of the domain. Thus on an infinite plane the second criterion is of no use. However, we could consider, for the sake of argument, how small the domain would have to be for the isolated vortices to be stable by this criterion. For instance, it turns out that for the cases shown in table 1, the boundary of the domain would have to be within the inner core of vorticity. This can be demonstrated by using a classical inequality relating the total energy in a closed domain to the total enstrophy (cf. Benzi, Pierini & Vulpiani 1982). Actually, the first criterion proves more useful here since it can be used to prove nonlinear stability for a boundary placed as far out as the edge of the inner core (by the monotonicity of the vorticity profile). We conclude that Arnol'd's method cannot even discern between isolated vortices that are known to be linearly stable and unstable ones. More general considerations where $V(r)$ may change sign lead to similar conclusions.

Appendix B. Calculation of the first and second variations

In order to calculate the changes at first and second orders of the various integral quantities of interest (energy, stream function, linear impulse, angular momentum), we have to determine in general how an integral of the form

$$I(\Gamma) = \iint_{\mathscr{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

changes when the boundary of the domain $\mathscr{D}$ is perturbed. Here $\mathscr{D}$ is a simply connected domain with a closed curve Γ as boundary, and $f(x,y)$ is some function. Let us write

$$\iint_{\mathscr{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathscr{D}} \tfrac{1}{2}\nabla\cdot\boldsymbol{F}\,\mathrm{d}x\,\mathrm{d}y = \tfrac{1}{2}\oint_{\Gamma} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}l, \tag{B 1}$$

where
$$\boldsymbol{F}(x,y) = \boldsymbol{i}\int^{x} f(u,y)\,\mathrm{d}u + \boldsymbol{j}\int^{y} f(x,u)\,\mathrm{d}u, \tag{B 2}$$

and
$$\boldsymbol{n} = \frac{\dot{\boldsymbol{s}}}{|\dot{\boldsymbol{s}}|}\wedge\boldsymbol{k}, \quad \mathrm{d}l = |\dot{\boldsymbol{s}}|\,\mathrm{d}\theta. \tag{B 3}$$

By substitution we find

$$I(\Gamma) = \tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \boldsymbol{F}(\boldsymbol{s})\wedge\dot{\boldsymbol{s}}\,\mathrm{d}\theta. \tag{B 4}$$

If Γ is changed, the integral becomes

$$I(\Gamma+\delta\Gamma) = \tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \boldsymbol{F}(\boldsymbol{s}+\delta\boldsymbol{s})\wedge(\dot{\boldsymbol{s}}+\delta\dot{\boldsymbol{u}})\,\mathrm{d}\theta,$$

where
$$\delta\boldsymbol{s} = \epsilon\boldsymbol{u}_1 + \tfrac{1}{2}\epsilon^2\boldsymbol{u}_2 + \dots .$$

Substituting in this expression the following Taylor-series expansion:

$$\boldsymbol{F}(\boldsymbol{s}+\delta\boldsymbol{s}) = \boldsymbol{F}(\boldsymbol{s}) + \epsilon\boldsymbol{u}_1\cdot\nabla\boldsymbol{F}(\boldsymbol{s}) + \tfrac{1}{2}\epsilon^2\{(\boldsymbol{u}_1\cdot\nabla)^2\boldsymbol{F}(\boldsymbol{r}) + \boldsymbol{u}_2\cdot\boldsymbol{F}(\boldsymbol{r})\}|_{\boldsymbol{r}=\boldsymbol{s}} + O(\epsilon^3),$$

and subtracting $I(\Gamma)$, the $O(\epsilon)$ and $O(\epsilon^2)$ changes are found to be given by

$$\begin{aligned} I(\Gamma+\delta\Gamma) - I(\Gamma) = {} & \epsilon\tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \{\boldsymbol{F}\wedge\dot{\boldsymbol{u}}_1\} + \{(\boldsymbol{u}_1\cdot\nabla\boldsymbol{F})\wedge\dot{\boldsymbol{s}}\}\,\mathrm{d}\theta \\ & + \epsilon^2\,\tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \{(\boldsymbol{u}_1\cdot\nabla\boldsymbol{F})\wedge\dot{\boldsymbol{u}}_1\} + \{(\tfrac{1}{2}(\boldsymbol{u}_1\cdot\nabla)^2\boldsymbol{F})\wedge\dot{\boldsymbol{s}}\}\,\mathrm{d}\theta \\ & + \epsilon^2\,\tfrac{1}{2}\boldsymbol{k}\cdot\int_{\theta_1}^{\theta_2} \{\tfrac{1}{2}\boldsymbol{F}\wedge\dot{\boldsymbol{u}}_2\} + \tfrac{1}{2}\{(\boldsymbol{u}_2\cdot\nabla\boldsymbol{F})\wedge\dot{\boldsymbol{s}}\}\,\mathrm{d}\theta + O(\epsilon^3), \end{aligned} \tag{B 5}$$

where all expressions involving $\boldsymbol{F}$ are understood to be evaluated at $\boldsymbol{r} = \boldsymbol{s}(\theta)$. The $O(\epsilon)$ change is deduced from this by noting that the integrand of the $O(\epsilon)$ variation can be written as

$$\boldsymbol{F} \wedge \dot{\boldsymbol{u}}_1 + (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \dot{\boldsymbol{s}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\{\boldsymbol{F} \wedge \boldsymbol{u}_1\} - (\dot{\boldsymbol{s}} \cdot \nabla \boldsymbol{F}) \wedge \boldsymbol{u}_1 + (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \dot{\boldsymbol{s}}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\{\boldsymbol{F} \wedge \boldsymbol{u}_1\} + (\nabla \cdot \boldsymbol{F})\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}}, \tag{B 6}$$

where we have used the identity

$$\{\boldsymbol{a} \cdot \nabla \boldsymbol{c}\} \wedge \boldsymbol{b} - \{\boldsymbol{b} \cdot \nabla \boldsymbol{c}\} \wedge \boldsymbol{a} = \{\nabla \cdot \boldsymbol{c}\}\, \boldsymbol{a} \wedge \boldsymbol{b}. \tag{B 7}$$

Because $\nabla \cdot \boldsymbol{F} = 2f$, we find

$$I_1 = \left.\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \tfrac{1}{2}\boldsymbol{k} \cdot \int_{\theta_1}^{\theta_2} \frac{\mathrm{d}}{\mathrm{d}\theta}\{\boldsymbol{F} \wedge \boldsymbol{u}_1\} + (\nabla \cdot \boldsymbol{F})\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}}\, \mathrm{d}\theta$$

$$= \boldsymbol{k} \cdot \int_{\theta_1}^{\theta_2} f(\boldsymbol{s})\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}}\, \mathrm{d}\theta, \tag{B 8}$$

where the contribution to the integral by the derivative is zero because of the periodicity on the closed curve.

The $O(\epsilon^2)$ variation in (B 5) can be cast in a more convenient form by rewriting the first part containing only $\boldsymbol{u}_1$ as

$$(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \dot{\boldsymbol{u}}_1 + (\tfrac{1}{2}(\boldsymbol{u}_1 \cdot \nabla)^2 \boldsymbol{F}) \wedge \dot{\boldsymbol{s}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\tfrac{1}{2}\{(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \boldsymbol{u}_1\} + \tfrac{1}{2}(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \dot{\boldsymbol{u}}_1 - \tfrac{1}{2}(\dot{\boldsymbol{u}}_1 \cdot \nabla \boldsymbol{F}) \wedge \boldsymbol{u}_1$$

$$-\tfrac{1}{2}\dot{\boldsymbol{s}} \cdot \nabla\{(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \boldsymbol{u}_1\} + \tfrac{1}{2}(\boldsymbol{u}_1 \cdot \nabla)^2 \boldsymbol{F} \wedge \dot{\boldsymbol{s}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\{\tfrac{1}{2}(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) \wedge \boldsymbol{u}_1\}$$

$$+\tfrac{1}{2}\{\nabla \cdot (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F})\}\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}} + \tfrac{1}{2}(\nabla \cdot \boldsymbol{F})\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{u}}_1, \tag{B 9}$$

where we have used (B 7) twice. Since $\nabla \cdot \boldsymbol{F} = 2f$ and $\nabla \cdot (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{F}) = 2\boldsymbol{u}_1 \cdot \nabla f$, the integrand of the first part is

$$(\boldsymbol{u}_1 \cdot \nabla f)\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}} + f \boldsymbol{u}_1 \wedge \dot{\boldsymbol{u}}_1.$$

The second part of the $O(\epsilon^2)$ variation in (B 5) containing $\boldsymbol{u}_2$ is of the same form as the first variation and we find therefore that

$$\{\tfrac{1}{2}\boldsymbol{F} \wedge \dot{\boldsymbol{u}}_2\} + \tfrac{1}{2}\{(\boldsymbol{u}_2 \cdot \nabla \boldsymbol{F}) \wedge \dot{\boldsymbol{s}}\} + \; = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\{\boldsymbol{F} \wedge \boldsymbol{u}_2\} + \tfrac{1}{2}(\nabla \cdot \boldsymbol{F})\, \boldsymbol{u}_2 \wedge \dot{\boldsymbol{s}}.$$

Putting everything together we find that the second variation is

$$I_2 = \left.\frac{\mathrm{d}^2 I}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} = \boldsymbol{k} \cdot \int_{\theta_1}^{\theta_2} (\boldsymbol{u}_1 \cdot \nabla f)\, \boldsymbol{u}_1 \wedge \dot{\boldsymbol{s}} + f\{\boldsymbol{u}_1 \wedge \dot{\boldsymbol{u}}_1 + \boldsymbol{u}_2 \wedge \dot{\boldsymbol{s}}\}\, \mathrm{d}\theta. \tag{B 10}$$

Appendix C. Conservation of E_2 and L_2

In this Appendix we show that the linearized dynamics conserve E_2 and L_2. We will only treat the case of the circular vortex consisting of n rings of different vorticities. First we note that the convention used in this paper relates the radial velocity v_r and azimuthal velocity v_θ (in polar coordinates (r,θ)) to the stream function ψ according to

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}.$$

The rate of change of the radial position of an element that has undergone a displacement $\epsilon\delta r_i + O(\epsilon^2)$ from the circle $r = d_i =$ constant, is given by

$$\frac{\partial(\epsilon\delta r_i + O(\epsilon^2))}{\partial t} = \frac{1}{d_i + \epsilon\delta r_i + O(\epsilon^2)}\frac{\partial}{\partial \theta}\Psi(d_i + \epsilon\delta r_i + O(\epsilon^2), \theta, t), \tag{C 1}$$

where t denotes time and Ψ is the stream function, which is developed in powers of the small-amplitude parameter

$$\Psi = \psi_0 + \epsilon\psi_1 + \tfrac{1}{2}\epsilon^2\psi_2 + \dots.$$

By expanding all terms in (C 1) we find the following $O(\epsilon)$ equation:

$$\frac{\partial \delta r_i}{\partial t} = \frac{1}{d_i}\left\{\frac{\mathrm{d}\psi_0}{\mathrm{d}r}(d_i)\frac{\partial \delta r_i}{\partial \theta} + \frac{\partial \psi_1}{\partial \theta}\right\},$$

which we write as

$$d_i\frac{\partial \delta r_1}{\partial t} = \frac{\partial \psi_1}{\partial \theta} - v_\theta(d_i)\frac{\partial \delta r_i}{\partial \theta}. \tag{C 2}$$

The first-order correction of the stream function (ψ_1) is calculated according to (79). With this we can now easily prove that E_2 and L_2, as given by (80) and (75), respectively, are invariants for the linearized equations of motion.

Let us first consider the second variation of the energy, which is

$$\tfrac{1}{2}E_2 = \int_0^{2\pi} \sum_i \Delta q_i \frac{v_\theta(d_i)}{d_i} d_i^2\, \delta r_i^2(\theta,t)\,\mathrm{d}\theta = \int_0^{2\pi} \sum_i \Delta q_i\, \psi_1(d_i,\theta,t)\, d_i\, \delta r_i(\theta,t)\,\mathrm{d}\theta. \tag{C 3}$$

Here Δq_i is defined by (76). The time derivative is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}E_2 &= \int_0^{2\pi} \sum_i 2\Delta q_i \frac{v_\theta(d_i)}{d_i} d_i^2\, \delta r_i \frac{\partial \delta r_i}{\partial t}\mathrm{d}\theta - \int_0^{2\pi} \sum_i 2\Delta q_i\, d_i\, \psi_1 \frac{\partial \delta r_i}{\partial t}\mathrm{d}\theta \\ &= \int_0^{2\pi} -\sum_i \Delta q_i \{\psi_1 - v_\theta(d_i)\}\, d_i \frac{\partial \delta r_i}{\partial t}\mathrm{d}\theta,\end{aligned} \tag{C 4}$$

where we have used the fact that

$$\int_0^{2\pi} \sum_i \Delta q_i\, d_i \frac{\partial \psi_1}{\partial t}\delta r_i\,\mathrm{d}\theta = \int_0^{2\pi} \sum_i \Delta q_i\, d_i\, \psi_1 \frac{\partial \delta r_i}{\partial t}\mathrm{d}\theta, \tag{C 5}$$

which follows by taking the time derivative of (79) and using it in the left-hand side of the above equation. Substitution of (C 2) in (C 4) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}E_2 = \int_0^{2\pi} -\sum_i \Delta q_i \frac{\partial}{\partial \theta}\{\psi_1 - v_\theta(d_i)\,\delta r_i\}^2\,\mathrm{d}\theta = 0. \tag{C 6}$$

The time derivative of (75) is

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = -4\int_0^{2\pi} \sum_i \Delta q_i\, d_i^2\, \delta r_i \frac{\partial \delta r_i}{\partial t}\mathrm{d}\theta, \tag{C 7}$$

which by using (C 2) becomes

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = -4\int_0^{2\pi} \sum_i \Delta q_i \left\{\frac{\partial \psi_1}{\partial \theta} - v_\theta(d_i)\frac{\partial \delta r_1}{\partial \theta}\right\} d_i\,\delta r_i\,\mathrm{d}\theta = -4\int_0^{2\pi} \sum_i \Delta q_i \frac{\partial \psi_1}{\partial \theta} d_i\,\delta r_i\,\mathrm{d}\theta. \quad \text{(C 8)}$$

By substitution of (79) and partial integration, the time derivative of L_2 is cast in the following form:

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = \frac{4}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sum_i \sum_j \Delta q_i\,\Delta q_j \log|d_i\,\mathrm{e}^{\mathrm{i}\theta} - d_j\,\mathrm{e}^{\mathrm{i}\theta'}|\,d_i \frac{\partial \delta r_i}{\partial \theta}(\theta, t)\,d_j\,\delta r_j(\theta', t)\,\mathrm{d}\theta\,\mathrm{d}\theta'.$$

With some elementary calculations this in turn is found to be equal to

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = -2\sum_i \sum_j \Delta q_i\,\Delta q_j\,d_i\,d_j \sum_{k=1}^{\infty} \Lambda^k_{i,j}(-a_{i,k}\,b_{j,k} + a_{j,k}\,b_{i,k}). \quad \text{(C 9)}$$

The $a_{i,k}$ and $b_{i,k}$ are the Fourier coefficients of the quantities $d_i\,\delta r_i$ (see (86)). Owing to its antisymmetry in $\{i,j\}$ this expression is clearly zero. This proves the time invariance of L_2.

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